SINGULAR SOBOLEV CONNECTIONS WITH HOLOMONY

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We consider local Sobolev connections on $SU(2)$ bundles over the complement, in $R^4$, of a smoothly embedded compact 2-manifold. Finite action implies that a holonomy condition is satisfied and we obtain an a priori estimate for the connection 1-form in terms of curvature and the flat connection carrying the holonomy. The a priori estimate classifies the possible singularities in these connections by the set of flat connections. In a certain case, this leads to smoothness and extendability results.

Let $N$ be a full 4-dimensional neighborhood of the singular set $S$. The objects of study are connections $D = d + A$ defined on $SU(2)$ bundles over $X = N\backslash S$. We assume that $A \in H^2_{1,\text{loc}}(X)$ and that the action is finite, i.e., the curvature $F = dA + A \wedge A$ is in $L^2(N)$.

The following holonomy condition was first stated by Cliff Taubes. Choose coordinates $(r, \theta, u, v)$ with $(u, v)$ coordinates on $S$ and $(r, \theta)$ coordinates in a plane normal to $S$. Fixing $u$ and $v$, and denoting by $A_\theta$ the $\theta$ component of $A$, the initial value problem for an $SU(2)$ valued function,

$$\frac{dg_r}{d\theta} + A_\theta g_r = 0, \quad g_r(0) = I,$$

has a unique solution $g_r(\theta)$, with $g_r(2\pi) = J_r \in SU(2)$. The holonomy condition we require is

(H) \quad \lim_{r \to 0} J_r = J^b \text{ exists.}

This condition is gauge invariant up to conjugacy in $SU(2)$. Our results can be formulated in two theorems.

**Theorem 1.** If $A$ and $F$ are smooth on $N\backslash S$ and $F \in L^2(N)$, then (H) is satisfied for almost all $u$ and $v$. Up to conjugacy, the limit is independent of $u$ and $v.

Next, assume (H) holds. Locally, the conjugacy class $[J^b]$ defines a flat connection $A^b = C d\theta$ with $C$ a constant element of $su(2)$ determined up to a similarity transformation. Our second result uses holonomy to obtain an a priori estimate. We denote by $X_0$ and $N_0$ the intersections of $X$ and $N$ with a small open set in $R^4$ having nonvoid intersection with $S$.

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THEOREM 2. Suppose \( \hat{D} = d + \hat{A} \) satisfies (H) with \( \hat{A} \in H^2_{1, \text{loc}}(X_0) \) and \( \|F\|_{L^2(N_0)} \) sufficiently small. Then there is a flat connection \( A^b \) determined by \([J^b]\), and a universal constant \( K \), such that \( \hat{D} \) is gauge equivalent to \( D = d + A \), with \( d^* A = 0 \) and

\[
\|A - A^b\|_{H^2(N_0)} \leq K\|F\|_{L^2(N_0)}.
\]

Note that if \([J^b] = I\), then \( A \) is gauge equivalent to the zero connection form. In this case, \( D \) extends as an \( H^2 \) connection to all of \( N_0 \). If, in addition, field equations are satisfied, more smoothness follows from elliptic theory.

Theorem 1 is proved by making a good choice of gauge in which the Fourier coefficients of \( A_\theta \) can be estimated in terms of \( F \). These estimates can be used to show that \( A_\theta d\theta \) converges to a flat connection as \( r \) tends to zero. This flat connection carries the holonomy. To show that the limit is independent of \( u \) and \( v \) requires another good choice of gauge and Stokes' theorem.

To prove Theorem 2, we carry out a plan of attack suggested by Cliff Taubes. This involves an open-closed argument similar to that used in [U₁, Theorem 1.3]. The large space consists of the appropriate Sobolev space of connections satisfying the same holonomy condition. This space is shown to be connected. The subspace consists of connections which admit a Hodge gauge satisfying certain boundary and limiting conditions which imply the a priori estimate. (Detailed proofs will appear in a forthcoming article.)

Theorem 1 settles a conjecture of Atiyah's. Both theorems are related to recent work on the moduli space of magnetic monopoles over hyperbolic 3-space \([A, B, C, F]\) and to Yang-Mills fields over \( S^4 \) whose topological charge is not integral \([FH_1, FH_2]\).

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REFERENCES


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