Multivariable Toeplitz operators, acting on Hardy or Bergman spaces over domains in $\mathbb{C}^n$, occur in connection with elliptic boundary value problems [1], weighted shift operators [6] and problems in function theory of several complex variables [2]. If the underlying domain is strictly pseudoconvex [4], of finite type [1, 11] or symmetric [13], the associated Toeplitz operators (with continuous symbol) are essentially commutative or at least generate a solvable $C^*$-algebra of finite length. In particular, the Toeplitz $C^*$-algebra is of type I.

In this note we describe the Toeplitz $C^*$-algebra of pseudoconvex Reinhardt domains $\Omega$, using a finite composition series which is geometrically characterized by “boundary foliations” associated with the complex geometry of $\Omega$. Whenever these foliations are of “irrational type,” we obtain Toeplitz $C^*$-algebras which are not of type I (this can happen for domains with smooth boundary). We also announce an index theory for these non-type I Toeplitz $C^*$-algebras and give some applications to the theory of proper holomorphic mappings. For concreteness, we explain here the case $n = 2$.

Let $\Omega$ be a bounded pseudoconvex complete Reinhardt domain (in $\mathbb{C}^2$), with closure $\overline{\Omega}$. By [8], these domains are the natural domains of convergence of power series and are characterized by the condition that $(u, v) \in \Omega$ whenever $|u| \leq |z|$, $|v| \leq |w|$ for some $(z, w) \in \Omega$ or $|u| = |z_1|^\lambda |w_1|^{1-\lambda}$, $|v| = |z_2|^\lambda |w_2|^{1-\lambda}$ for some $(z_1, w_1) \in \Omega$, $(z_2, w_2) \in \Omega$ and $0 < \lambda < 1$. We may assume that $\Omega$ is normalized, i.e., $\Omega$ is contained in the bidisk $\mathbb{D}^2$ and contains the coordinate axes $V := \{(z, w) \in \mathbb{D}^2: zw = 0\}$. Then the “logarithmic domain” $C := \{(x, y) \in \mathbb{R}^2: (e^x, e^y) \in \Omega\}$ is an unbounded convex open set contained in the third quadrant and $\partial C$ is a concave curve. Let $\overline{C}$ denote the closure of $C$ in $\mathbb{R}^2$ and let $\partial^j(C)$ be the union of all $j$-dimensional faces of $\overline{C}$ (e.g., $\partial^2(C) = \overline{C}$ and $\partial^0(C)$ consists of all extreme points).

Given a face $F$ of $\overline{C}$, denote by $L_F$ the linear subspace of the same dimension parallel to $F$. For any point $t = (\xi, \eta)$ in the 2-torus $T^2$, consider the leaf $t_F := \{(\xi e^{2\pi i x}, \eta e^{2\pi i y}): (x, y) \in L_F\}$ generated by $F$ through $t$. This gives a foliation $\mathcal{F}_F$ of $T^2$, with corresponding foliation $C^*$-algebra (cf. [5]) denoted by $C^*(\mathcal{F}_F)$. For $F = \overline{C}$, $\mathcal{F}_F$ has just one leaf ($T^2$ itself) and $C^*(\mathcal{F}_F)$ is *-isomorphic to the ideal $\mathcal{H}$ of compact operators. For
$F = P$, an extreme point in $\partial C, \mathcal{F}_F$ is the trivial foliation where every point of $T^2$ is a leaf, and $C^*(\mathcal{F}_F) \cong C(T^2)$. Here $C(X)$ is the $C^*$-algebra of all continuous functions on a compact space $X$. If $F$ is one-dimensional, $\mathcal{F}_F$ is the foliation of the Kronecker flow determined by the slope of $F$.

Let $H^2(\Omega)$ be the Bergman space of all (Lebesgue) square integrable holomorphic functions on $\Omega$. Let $P : L^2(\Omega) \to H^2(\Omega)$ be the (orthogonal) Bergman projection. Then, for every $\varphi \in C(\Omega)$, the bounded operator $T_{\varphi}$ on $H^2(\Omega)$ defined by

$$T_{\varphi}(f) := P(\varphi f), \quad f \in H^2(\Omega)$$

is called the Toeplitz operator with symbol $\varphi$. The $C^*$-algebra generated by all these operators is denoted by $\mathcal{T}(\Omega)$.

**Theorem 1.** Let $\Omega$ be a (normalized) pseudoconvex complete Reinhardt domain in $\mathbb{C}^2$. Then the Toeplitz $C^*$-algebra $\mathcal{T}(\Omega)$ has a composition series $\mathcal{H} \subset \mathcal{F} \subset \mathcal{T}(\Omega)$, where $\mathcal{F}$ is the commutator ideal,

$$\mathcal{T}(\Omega)/\mathcal{F} \cong C(\partial^0(\Omega))$$

and

$$\mathcal{F}/\mathcal{H} \cong \sum_F C^*(\mathcal{F}_F) \quad (C^*\text{-algebraic sum}).$$

Here $\partial^0(\Omega)$ is the closure (in $\mathbb{C}^2$) of the set $\{(xe^x, ye^y) : (x, y) \in T^2, (x, y) \in \partial^0(C)\}$ and $F$ runs over all 1-dimensional faces of $\overline{C}$.

If we let $\mathcal{H}_0 = 0$, $\mathcal{H}_1 = \mathcal{H}$, $\mathcal{H}_2 = \mathcal{F}$ and $\mathcal{H}_3 = \mathcal{T}(\Omega)$, we can uniformly state the conclusion of Theorem 1 as

$$\mathcal{H}_{j+1}/\mathcal{H}_j \cong \int_F C^*(\mathcal{F}_F) \quad (C^*\text{-direct integral}),$$

where $F$ runs over all $(2 - j)$-dimensional faces of $\overline{C}$, $0 \leq j \leq 2$.

**Corollary 2.** $\mathcal{T}(\Omega)$ is of type I if and only if the slope of every 1-dimensional face in $\partial^1(C)$ is rational. Further, $\mathcal{T}(\Omega)$ is essentially abelian, i.e., $\mathcal{F} = \mathcal{H}$, if and only if there is no 1-dimensional face in $\partial C$, i.e., $\partial^1(C) = \emptyset$.

The above results are proved in detail in [12]. The following purely geometrical result is a direct consequence of Corollary 2 and [11, Corollary 3.2].

**Corollary 3.** Let $\Omega$ and $\Omega'$ be two normalized pseudoconvex complete Reinhardt domains. Let $C$ and $C'$ be the corresponding logarithmic domains, and assume there is a proper holomorphic mapping $\varphi : \Omega \to \Omega'$. If $\partial C$ contains no 1-dimensional faces with irrational slope, then the same property holds for $\partial C'$. Further, if $\partial C$ contains no 1-dimensional faces, then the same is true for $\partial C'$.

Now we describe the index phenomenon in the presence of irrational slopes. We do this in the simplest nontrivial case, i.e., when $\Omega$ is the logarithmic convex hull of the union of two polydisks of multiradii $(e, 1)$
and \((1, \delta), \varepsilon < 1, \delta < 1\) (cf. [6]). Then the boundary of \(C\) consists of the line segment \(F\) joining \((\log \varepsilon, 0)\) and \((0, \log \delta)\) together with the negative part of both axes between \((-\infty, 0)\) and \((\log \varepsilon, 0)\) and between \((0, -\infty)\) and \((0, \log \delta)\). Assume that the corresponding slope \(\beta = -\log \delta / \log \varepsilon\) is irrational. Then, as a consequence of Theorem 1, we have

\[
\mathcal{J} / \mathcal{H} \cong [\mathcal{C}(T) \otimes \mathcal{H}] \oplus [\mathcal{C}(T) \otimes \mathcal{H}] \oplus C^*(\mathcal{F})
\]

and

\[
\mathcal{I} (\Omega) / \mathcal{I} \cong \mathcal{C}(T^2) \oplus \mathcal{C}(T^2).
\]

Let \(Z^2\) act on \(\mathbb{R}\) by \(\alpha(m, n; x) = x - n - m \beta^{-1},\) for \(x \in \mathbb{R}\) and \((m, n) \in Z^2\). The associated (strongly continuous) action of \(Z^2\) on \(\mathbb{R}(\mathbb{R})\), again denoted by \(\alpha\), induces a crossed product \(C^*\)-algebra \(\mathbb{R}(\mathbb{R}) \rtimes \alpha Z^2\) (defined as the \(C^*\)-completion of the convolution algebra of \(\mathbb{R}(\mathbb{R})\)-valued \(L^1\)-functions on \(Z^2\), cf. [9]), which is isomorphic to \(C^*(\mathcal{F})\) (not just stably isomorphic, cf. [10]). Further, we have \(C^*(\mathcal{F}) \cong A_\beta \otimes \mathcal{H}\), where \(A_\beta := \mathcal{C}(T) \rtimes \beta\) is the \textit{irrational rotation \(C^*\)-algebra} induced by the action of \(Z\) on \(T\) generated by the rotation with angle \(\beta\). By Theorem 1, there is an ideal \(\mathcal{I}_{\text{sing}} \subset \mathcal{I}\) containing \(\mathcal{H}\) such that \(\mathcal{I}/\mathcal{I}_{\text{sing}} \cong [\mathcal{C}(T) \oplus \mathcal{C}(T)] \otimes \mathcal{H}\) and \(\mathcal{I}/\mathcal{I}_{\text{sing}} \cong \mathbb{R}(\mathbb{R}) \rtimes \alpha Z^2\). The ideal \(\mathcal{I}_{\text{sing}}\) induces an exact sequence

\[
0 \to \mathcal{I} / \mathcal{I}_{\text{sing}} \to \mathcal{I}(\Omega) / \mathcal{I}_{\text{sing}} \to \mathcal{I}(\Omega) / \mathcal{I} \to 0,
\]

where \(\mathcal{I}(\Omega) / \mathcal{I}_{\text{sing}} \cong \mathcal{C}(\mathbb{R} \cup \{\pm \infty\}) \rtimes \alpha Z^2\). Any short exact sequence \(0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0\) of \(C^*\)-algebras has a topological invariant called the \textit{index mapping} \(\text{Ind}: K_1(\mathcal{A}) \to K_0(\mathcal{A})\) on the level of \(K\)-theory (cf. [3]), which reduces to the ordinary (family) Fredholm index in case \(\mathcal{A} = \mathcal{H}\) and \(\mathcal{C}\) is commutative.

**Theorem 4.** The \textit{analytical index map}

\[
\text{Ind}: K^1(T^2) \oplus K^1(T^2) \to K_0(C^*(\mathcal{F})),
\]

associated with the above exact sequence (cf. [7]) has the topological expression

\[
\text{tr}(\text{Ind}(\phi \oplus \psi)) = \alpha(\text{ch}(\phi \psi^{-1}); 0) \quad \text{for } \phi, \psi \in K^1(T^2),
\]

where \(\text{tr}: K_0(C^*(\mathcal{F})) \to \mathbb{R}\) is the natural trace and

\[
\text{ch}: K^1(T^2) \to H^1(T^2, \mathbb{Z}) \cong \mathbb{Z}^2
\]

is the classical Chern character.

For the proof of the above theorem, see [12].

**Remark 5.** We can easily construct a continuous function \(\theta\) on \(\overline{\mathbb{R}}\) such that the above index applied to the image of \(T_\theta\) in \(\mathcal{I}(\Omega) / \mathcal{I}\) yields a nonzero irrational number. For instance, let \(\theta\) be any continuous function on \(\overline{\mathbb{R}}\) such that \(\theta(z, w) = w\) for \(|z| = \varepsilon, |w| = 1\), and such that \(\theta(z, w) = z\), for \(|z| = 1, |w| = \delta\).
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