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Real variable methods in harmonic analysis, by Alberto Torchinsky. Pure and Applied Mathematics, vol. 123, Academic Press, Inc., Orlando, 1986, xii + 462 pp., $89.00 ($45.00 paper). ISBN 0-12-695460-7

Weighted norm inequalities and related topics, by J. García-Cuerva and J. L. Rubio de Francia. Mathematical Studies, vol. 116, North-Holland, Amsterdam, New York, Oxford, 1985, x + 604 pp., ISBN 0-444-87804-1

In the preface of their book, García-Cuerva and Rubio de Francia say: "After the classical monograph of A. Zygmund [**31**], the standard references for the important developments that occurred in Fourier Analysis during the second half of this century are E. M. Stein [**25**] and Stein and Weiss [**29**], both published around 1970." Since that date there have been dramatic advances in several areas of Harmonic Analysis. We shall describe some of these. We begin with the **Theory of Hardy spaces** and shall present a more elaborate description of this development since we can use some of the material presented to help explain the progress made in other areas.

Classical Harmonic Analysis in one dimension is either associated with the Real line $\mathbf{R}$ or the **Torus** $\mathbf{T} = [0, 2\pi)$, often identified with the **Circle Group** $\{z \in \mathbf{C}: z = e^{i\theta},\ 0 \le \theta < 2\pi\}$. Let us concentrate on $\mathbf{R}$, which we consider embedded in $\mathbf{R}^2$ (or $\mathbf{C}$) as the boundary of the **Upper Half Plane** $\mathbf{R}^2_+ \equiv \{z = (x, y) \in \mathbf{R}^2: y > 0\} = \{x + iy \in \mathbf{C}: y > 0\}$. If $0 < p \le \infty$ the **Hardy Space** H^p consists of all holomorphic functions $F(x + iy)$ on $\mathbf{R}^2_+$ such that

$$\|F\|_{H^p} \equiv \sup_{y>0} \left\{ \int_{-\infty}^{\infty} |F(x + iy)|^p \, dx \right\}^{1/p} < \infty. \tag{1}$$

These spaces are related to the Lebesgue spaces $L^p(\mathbf{R})$ via the following theorems:

EXISTENCE OF BOUNDARY VALUES. *If* $F \in H^p$ *then* $\lim_{y\to 0} F(x+iy) \equiv F(x)$ *exists for almost every* $x \in \mathbf{R}$. *Moreover, if* $0 < p < \infty$, *then* $\lim_{y\to 0} \int_{-\infty}^{\infty} |F(x+iy) - F(x)|^p\, dx = 0$.

When $0 < p$ the real part, $u(z)$, of $F(z)$ converges, as $y \to 0$, to a real valued function $f(x)$ on $\mathbf{R}$ that belongs to L^p. When $1 < p < \infty$ one can obtain a converse result: if f is a real valued function in L^p then the function on $\mathbf{R}^2_+$ defined by

$$F(z) = u(z) + iv(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{z-t}\, dt \tag{2}$$

belongs to H^p, the real part of its boundary value equals $f(x)$ a.e. and the mapping $f \to F$ is a bounded linear transformation.

THE M. RIESZ INEQUALITY. *If* $f \in L^p(\mathbf{R})$, $1 < p < \infty$, *and* F *is given by equality* (2) *then* $\|F\|_{H^p} \le A_p \|f\|_p$, *where* A_p *is independent of* f.

Thus, for this range of p, one can assign to each $f \in L^p(\mathbf{R})$ the H^p-norm of the associated function F (via equality (2)) and one obtains a norm equivalent to the L^p-norm. This is sometimes stated simply by the assertion that "real H^p equals real L^p when $1 < p < \infty$." When $p = 1$ there exist real valued functions L^1 that are not the real part of boundary values of an $F \in H^1$. That is, "real H^1 is a proper subspace of real L^1." This is also reflected by the fact that the constant A_p in the M. Riesz inequality behaves like $1/(p-1)$ as $p \to 1$.

When $0 < p < 1$ the situation is more complicated. Even though the boundary values $F(x)$ exist a.e., the real parts $f(x)$ no longer determine the holomorphic function F on $\mathbf{R}^2_+$. The proper interpretation of these boundary values involves distributions and not functions; thus, the space "real H^p" for $p < 1$ is an appropriate space of (tempered) distributions. It is not a Banach space since the functional in (1) is not a norm; however, it is a "quasinorm" (Minkowski's inequality is replaced by a similar one: $\|a+b\| \le K(\|a\| + \|b\|)$ with $K = K(p) > 1$) and it turns H^p into an interesting topological vector space that is not locally convex. For details about the classical theory of H^p spaces see **[31** and **16]**.

One can, however, consider many of these notions without using analytic functions. The boundary values of the imaginary part of an F in H^p can be expressed directly in terms of the **Hilbert Transform** of the boundary values of the real part f:

$$\tilde{f}(x) = (Hf)(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi}\int_{|t|>\varepsilon} \frac{f(x-t)}{t}\, dt \tag{3}$$

(formally, this is the integral in (2) with $\operatorname{Im} z = 0$). A reformulation of the M. Riesz inequality is that the limit defining $\tilde{f}$ exists a.e. when $f \in L^p$, $1 < p < \infty$, and is a bounded operator. This limit does exist when $p = 1$; however, it does not define a bounded operator on L^1. Real H^p, then, is the space of all those $f \in L^p$ whose Hilbert transform belongs to L^p.

When $1 < p < \infty$ these two spaces are the same; when $p = 1$, H^1 is a proper subspace of L^1.

Around 1950 there was a great impetuous to develop Fourier Analysis in higher dimensions. This was created in large part by Calderón and Zygmund through their study of singular integrals (which we shall describe below). The push was accompanied by extensions of the theory of Hardy spaces to several dimensions. There are several more or less obvious settings for such generalizations: (i) the unit disk can be replaced by its n-fold Cartesian product, **the Polydisc**. Holomorphic functions of one variable are then replaced by holomorphic functions of several variables and Fourier series are replaced by multiple Fourier series; (ii) the unit disk can be replaced by the unit ball in $\mathbf{C}^n$. Again, Hardy space theory involves holomorphic functions of several complex variables; the boundary of the ball, however, does not enjoy some of the features of the circle group (when $n = 2$ this boundary can be identified with the group SU(2); but it is not a group when $n > 2$); (iii) in the noncompact case there are parallel directions. The n-fold Cartesian product of $\mathbf{R}$ leads to a theory similar to the one associated with the polydisc. There are many other directions and approaches that have been developed in the twenty year period following 1950. Some of these are considered in the articles [**7**, **30** and **12**].

There is one n-dimensional extension of H^p theory, however, that in addition to being natural, fits in particularly well with later developments. It is based on the following elementary facts about analytic functions of a complex variable. Suppose that $F = u + iv$ is such a function defined on a simply connected domain $\Omega \subset \mathbf{C}$. Then the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ tell us, first, that (v, u) is the gradient of a function h on Ω and, second, that $\Delta h = 0$ (that is, that h is harmonic). Thus, one can consider gradients of harmonic functions on a domain $\Omega \subset \mathbf{R}^n$ to be extensions of the notion of a holomorphic function on a domain in $\mathbf{R}^2 (= \mathbf{C})$. In view of this observation and keeping in mind the definition of Hardy spaces induced by condition (1) we can consider the following spaces to be an extension of the spaces H^p to n dimensions. Let $F = (v_1, v_2, \ldots, v_n, u) = (V, u)$ be a mapping from the **Upper Half Space** $\mathbf{R}^{n+1}_+ \equiv \{(x_1, x_2, \ldots, x_n, y) = (x, y) \in \mathbf{R}^{n+1} : y > 0\}$ onto $\mathbf{R}^{n+1}$ satisfying the **Generalized Cauchy-Riemann Equations**

$$(4) \qquad \text{(i)} \ \frac{\partial u}{\partial y} + \sum_1^n \frac{\partial v_j}{\partial x_j} = 0, \quad \text{(ii)} \ \frac{\partial v_j}{\partial x_k} = \frac{\partial v_k}{\partial x_j}, \qquad \frac{\partial v_j}{\partial y} = \frac{\partial u}{\partial x_j}$$

for $j, k = 1, 2, \ldots, n$. We say that such an F belongs to the space $H^p(\mathbf{R}^n)$, $p > 0$, if and only if

$$(5) \qquad \|F\|_{H^p} \equiv \sup_{y>0} \left\{ \int_{\mathbf{R}^n} |F(x, y)|^p \, dx \right\}^{1/p} < \infty.$$

These spaces were introduced in [**28**]; there it was shown that many of the properties of the classical Hardy spaces are valid for these n-dimensional extensions. For example, the boundary values

$$F(x) = (v_1(x), v_2(x), \ldots, v_n(x), u(x)) = \lim_{y \to 0} F(x, y)$$

exist for almost every $x \in \mathbf{R}^n$ as well as in the L^p norm, provided $p > (n-1)/n$ (for lower values of the index p, one has to consider more general versions of the Cauchy-Riemann equations [**28**, **6**, **29**]). As is the case when $1 < p < \infty$, the boundary function $u(x)$ is the general element of $L^p(\mathbf{R}^n)$ and it uniquely determines the boundary values of the "conjugates" $v_j(x)$,

$$v_j(x) = (R_j u)(x) = \lim_{\varepsilon \to 0} c_n \int_{|t|>\varepsilon} u(x-t) \frac{t_j}{|t|^{n+1}}\, dt. \tag{6}$$

R_j is a natural extension of the Hilbert transform (compare with (3)) and is called the jth **Riesz Transform** (it was M. Riesz who first considered this n-dimensional version of the Hilbert transform).

Though many other properties of the one dimensional spaces were extended to these higher dimensional analogs, some basic questions were not resolved. In fact, many were unresolved even in the one dimensional case. We have seen how the spaces H^p correspond to the boundary spaces L^p when $1 < p < \infty$. A natural problem is to characterize the subspace of $L^1(\mathbf{R})$ consisting of the real parts of boundary values of functions in $H^1(\mathbf{R}^2_+)$. Another question is: What is the dual of this subspace of L^1? Once these problems are resolved one can then consider their natural extension to $\mathbf{R}^n$, $n > 1$. In the decade 1970–1980 there occurred many major breakthroughs in the theory of Hardy spaces; in particular, the solutions of the above questions were obtained.

Burkholder, Gundy and Silverstein [**1**], using probability theory methods, characterized the harmonic functions $u(x, y)$ on $\mathbf{R}^2_+$ that are the real parts of functions in H^1. Let

$$u^*(x) = \sup_{|w-x|<y} |u(w, y)|.$$

They showed that u is such a function if and only if $u^* \in L^1(\mathbf{R})$. C. Fefferman and Stein [**18**] extended this result to n dimension for the Hardy spaces associated with $\mathbf{R}^{n+1}_+$ we described above. At just about the same time, C. Fefferman identified the dual of H^1 with the space of functions of **Bounded Mean Oscillation** (BMO) that was introduced a few years earlier by John and Nirenberg [**21**]. Fefferman and Stein extended this result to the n-dimensional case. Appropriate versions of all these results extend to the case $p > 0$.

Another important breakthrough was the discovery by Coifman of the **Atomic** characterization of the spaces [**8**]. An **atom** is, simply, an $L^1(\mathbf{R})$ function $a(x)$ that is supported in a finite interval I, and satisfies the size and cancellation conditions

$$\text{(i) } |a(x)| \le 1/l(I) \quad \text{and} \quad \text{(ii) } \int a(x)\, dx = 0, \tag{8}$$

where $l(I)$ is the length of I. It is an easy exercise to show that the Hilbert transform $\tilde{a}(x)$ satisfies $\|\tilde{a}\|_1 \le 7$. Thus, an atom belongs to real H^1 and $\|a + i\tilde{a}\|_{H^1} \approx \|a\|_1$. It follows that any function of the form

$$f = \sum_1^\infty \lambda_n a_n, \tag{9}$$