CONJECTURE "EPSILON" FOR WEIGHT \( k > 2 \)

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Introduction. Let \( N, k \geq 1 \) be integers and suppose \( \varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times \) is a Dirichlet character. Set

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \mod N \right\}.
\]

A holomorphic function \( f \) on the Poincaré upper half plane is called a cusp form of level \( N \), weight \( k \), and Nebentypus \( \varepsilon \) (or briefly a cusp form of type \( (N, k, \varepsilon) \)) if \( f \) has a zero at each cusp and

\[
f \left( \frac{az + b}{cz + d} \right) = \varepsilon(d)(cz + d)^k f(z) \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).
\]

The space of all such cusp forms is denoted \( S_k(\Gamma_0(N), \varepsilon) \); this space is acted upon by the Hecke operators \( \{T_n, n > 1\} \). A function \( f \in S_k(\Gamma_0(N), \varepsilon) \) has a Fourier expansion

\[
f = \sum_{n \geq 1} a_n(f)q^n = \sum_{n \geq 1} a_nq^n, \quad q = e^{2\pi iz}.
\]

It is said to be normalized if \( a_1(f) = 1 \). A normalized \( f \in S_k(\Gamma_0(N), \varepsilon) \) which is an eigenfunction of all the Hecke operators has \( a_n = a_n(f) \) lying in the ring of integers \( \mathcal{O} = \mathcal{O}_f \) of the algebraic number field \( K_f = \mathbb{Q}(a_n)_{n \geq 1} \).

Consider now a continuous irreducible representation

\[
\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\overline{\mathbb{F}}_l)
\]

which is odd, i.e. \( \det(\rho(c)) = -1 \) where \( c \) is complex conjugation. Let \( f \in S_k(\Gamma_0(N), \varepsilon) \) be a normalized eigenfunction of the Hecke operators and \( \lambda \) be a prime above \( l \) in the extension of \( K_f \) generated by the values of \( \varepsilon \). Suppose that for each prime \( p \) unramified in \( \rho \)

\[
\text{Tr}(\rho(\text{Frob}_p)) = a_p(f) \mod \lambda
\]

\[
\det(\rho(\text{Frob}_p)) = \varepsilon(p)p^{k-1} \mod \lambda
\]

where \( \text{Frob}_p \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) is a Frobenius element at \( p \). Then we say that \( \rho \) arises from \( f \). In [Se 2], Serre conjectures that every such \( \rho \) arises from some cusp form \( f \). He furthermore gives a procedure for determining from \( \rho \) the type \( (N, k, \varepsilon) \) of a modular form which gives rise to it. A second conjecture found in [Se 1], implied by the conjecture above, asserts that in certain cases if such a \( \rho \) arises from a modular form then it arises from one of the predicted level. This second conjecture was isolated in the case of weight two and dubbed "Epsilon" because it was sufficient due

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to the construction of G. Frey to show that the Taniyama-Weil Conjecture implies Fermat's Last Theorem. K. Ribet subsequently proved "Epsilon," cf. [Ri]. His proof is based on the jacobians of Shimura and modular curves.

Both of the conjectures above can be viewed as first steps in articulating a "Langlands philosophy mod $p$" for the special case of the group $GL(2, \mathbb{Q})$. As such, a context other than jacobians must be found so that the settings of higher weight, other groups, etc. may be studied. It is the purpose of this note to outline the proof of a weight $k$ analogue of Conjecture "Epsilon." Perhaps the formulations offered here will also be of use in considering more general groups.

**Theorem 1.** Let $k \geq 4$ be an even integer and $p,l$ be distinct prime numbers, $l > k$. Suppose the irreducible representation $\rho: Gal(\mathbb{Q}/\mathbb{Q}) \to GL_2(\overline{\mathbb{F}}_l)$ arises from $S_k(\Gamma_0(pM))$ with $(p, M) = 1$. Assume $\rho$ is unramified at $p$. Then $\rho$ also arises from $\ast S^k(r_0(M))$.

**Remarks.** (1) We can treat odd weight greater than 1 and Nebentypus of conductor prime to $p$, but for simplicity this is not considered here.

(2) The restriction $l > k$ may be inessential to our proof, but we definitely need $l > 2$.

(3) Detailed proofs shall be forthcoming.

To prove Theorem 1, we follow closely the leitfaden of Ribet's proof [Ri] in the case of weight 2, showing that all ingredients may be supplied for weight $k$. This work was partially supported by the U.S.-Israel Binational Science Foundation, and the first author was also partially supported by the National Science Foundation. It is a pleasure to thank G. Faltings and K. Ribet for their generous help and interest.

I. Sheaves on graphs attached to definite quaternion algebras and equivariant cohomology [Jo-Li]. Let $\Delta$ be the Bruhat-Tits tree for $SL_2(\mathbb{Q}_p)$ with vertices $\text{Ver} \, \Delta$, and edges $\text{Ed} \, \Delta$. Let $D/\mathbb{Q}$ be a quaternion algebra of discriminant $q \infty$, with $q$ a prime, $q \neq p$. Fix an Eichler order $\mathcal{M} \subseteq D$ of conductor $M$, $(M, q) = 1$. Set $\overline{\Gamma} = \mathcal{M}[\frac{1}{p}]^\times$ and $\Gamma = \{ \gamma \in \overline{\Gamma} | N_{D/\mathbb{Q}}(\gamma) \text{ has even } p\text{-adic valuation} \}$, viewed as discrete co-compact subgroups of $GL_2(\mathbb{Q}_p)$. They act on $\Delta$ and $Y = \Gamma \backslash \Delta$ is a finite graph. Let $L$ be a $\mathbb{Z}[\frac{1}{p}]$-model of the representation $\text{Symm}^{k-2}$ of $\mathcal{M}[\frac{1}{p}]^\times$; $L$ is explicitly constructed in [Jo-Li].

For a prime $l \neq p$, set $L_l = L \otimes_{\mathbb{Z}[\frac{1}{p}]} \mathbb{Z}_l$. Regarding $L$ as a constant sheaf over $\Delta$ and letting $\pi: \Delta \to \Gamma \backslash \Delta$ be the projection, set $\mathcal{L} = (\pi_* L)^\Gamma$ and $\mathcal{L}_l = \mathcal{L} \otimes_{\mathbb{Z}[\frac{1}{p}]} \mathbb{Z}_l$. Finally let $H^*_e(\Delta, -)$ denote $\Gamma$-equivariant cohomology as in [Br].

We will now define the weight $k$ analogues of the duals to the character groups of the Néron special fibers of jacobians of Shimura and modular curves. Let $B$ be the indefinite rational quaternion algebra ramified at $p$ and $q$. Denote by $V_B(M)$ the Shimura curve attached to an Eichler order
of conductor $M$ in $B$. Make the following definitions:

$$\hat{X}_q(X_0(qM))_l \overset{\text{def}}{=} C^0_F(L_l) = \text{Hom}_F(\text{Ver} \Delta, L_l)$$

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$$\hat{X}_p(V_B(M))_l \overset{\text{def}}{=} H^1(Y, \mathcal{L}_l) = H^1_\Delta(\Delta, L_l).$$

All of the above modules are equipped with natural inner products. Moreover Hecke operators constructed from $D$ act. In addition, let $\alpha \in \mathcal{M}[\mathbb{Z}]$ be an element of norm $p$. Then $\alpha$ induces an operator $w_p$ on $\hat{X}_q(X_0(pqM))_l$ and $\hat{X}_q(V_B(M))_l$ with $w_p^2$ acting as multiplication by $p^{k-2}$. There is an isomorphism

$$\hat{X}_q(X_0(qM))_l^2 = C^0_F(L_l) \oplus C^0_F(L_l) \overset{\cong}{\to} C^0_F(L_l).$$

**Theorem 2 (The Ribet exact sequence).** There is an exact sequence

$$0 \to \hat{X}_q(X_0(qM))_l^2 \overset{\iota}{\to} \hat{X}_q(X_0(pqM))_l \to \hat{X}_p(V_B(M))_l \to 0$$

compatible with the action of Hecke (i.e., equivariant for Hecke operators $T(l)$, $l \neq p$, and known action for $T(p)$).

This Ribet Exact Sequence is simply the cellular resolution for equivariant cohomology

$$C^0_F(L_l) \xrightarrow{\delta} C^1_F(L_l) \to H^1_\Delta(\Delta, L_l) \to 0.$$  

There is a similar sequence of $\mathbb{Z}[\mathcal{M}]$-modules when we replace $L_l$ by $L$.

Higher weight analogues to the group of connected components of the Néron special fibers of jacobians of Shimura and modular curves can also be defined. For $r = p$ or $q$ and $C$ the curve $X_0(rM)$, $X_0(pqM)$ or $V_B(M)$, set

$$\Phi_r(C)_l = \hat{X}_r(C)_l/\hat{X}_r(C)_l^2.$$  

If $C$ is an elliptic modular curve, then $\Phi_r(C)_l = 0$ (recall that $l \neq 2, 3$). For Shimura curves these groups admit the following description. Let $X_l = \hat{X}_p(V_B(M))_l$. Then

$$\Phi_p(V_B(M)) \approx (X_l/(p^{(k-2)/2}(1+p) + T(p))X_l)$$

$$\oplus (X_l/(p^{(k-2)/2}(1+p) - T(p))X_l).$$

**Theorem 3.** $\Phi_p(V_B(M))_l$ is a module of fusion relative to the decomposition

$$S_k(\Gamma_0(pqM))^{q - \text{new}} = [S_k(\Gamma_0(qM))^{q - \text{new}}]^2 \oplus [(S_k(\Gamma_0(qM))^{q - \text{new}})^2]_l.$$  

**Corollary 4.** There are infinitely many primes $q \equiv 1 \mod l$ such that $\rho$ also comes from $f' \in S_k(\Gamma_0(pqM))^{q - \text{new}}$.

This “raising the level” result is proved precisely as in [Ri].
II. Sheaf cohomology on modular and Shimura curves. The following results were proved by one of us a year ago. For a natural number \( N \) let \( \phi: E \to Y_0(N) \) be the universal elliptic curve. Suppose \((l, N) = 1\) and let \( \partial_1 \) be the locally constant \( l \)-adic sheaf \( \text{Symm}^{k-2}(R^1\phi_*\mathbb{Z}_l) \) on \( Y_0(N) \). On \( V_B(M) \) with \((l, Mpq) = 1\) let \( \partial_1 \) be the sheaf \( \text{Symm}^{k-2}\mathcal{F}_1 \) where \( \mathcal{F}_1 \) is defined as in [Ca 1]. Equivalently \( \mathcal{F}_1 \) may be defined by an idempotent splitting \( \mathcal{F}_1 \oplus \mathcal{F}_2 \approx R^1\phi_*\mathbb{Z}_l \), where \( \phi': A \to V_B(M) \) is the universal abelian surface. While \( E \) and \( A \) are defined only locally for the étale topology, \( \partial_1 \) and \( \mathcal{F}_1 \) make sense.

THEOREM 5. Let \( \Phi_r(\overline{C})(l|(-1) \) be the \((-1)\)-Tate-twist of the kernel of multiplication by \( l \) on \( \Phi_r(\overline{C}) \). Then there is a Hecke and Gal\( (\overline{F}_r/F_r) \)-equivariant exact sequence

\[
0 \to H^1_{par}(C \times \overline{F}_r, \overline{\partial}_1) \to H^1_{par}(C \times \overline{Q}_r, \overline{\partial}_1)^l \to \Phi_r(\overline{C})(l|(-1) \to 0,
\]

where \( I \subset \text{Gal}(\overline{Q}_r/Q_r) \) is the inertia subgroup.

THEOREM 6. There is an exact sequence of \( T \)-modules

\[
0 \to \hat{X}_r(\overline{C}) \otimes (\mathbb{Z}/l\mathbb{Z}) \to H^1_{par}(C \times \overline{F}_r, \overline{\partial}_1) \to (H^1_{par})^{\text{old}} \to 0
\]

where \((H^1_{par})^{\text{old}} \) is 0 in the Shimura curve case and is isomorphic to two copies of \( H^1_{par}(Y_0(N/r), \overline{\partial}_1) \) when \( C = Y_0(N) \). Again, some care must be taken regarding the action of \( T_r \).

Define Frobenius at \( r \) to act on \( \hat{X}_r(\overline{C}) \) and \( \Phi(\overline{C}) \) via \( w_r \) if \( C \) is a Shimura curve and by \(-w_r \) if \( C \) is an elliptic modular curve. Then the above short exact sequence is Frobenius-equivariant.

For the proof one uses the end of [La] and §11 of [Ca 2].

III. Crystalline cohomology and Galois representations. Gerd Faltings has kindly explained to us how the relevant multiplicity 2 result can be deduced from applying his recent work on crystalline cohomology for open varieties with normal crossings compactifications [Fa 1] to the context studied in his previous work [Fa 2]. To set notation, let \( N \geq 1 \) be an integer, let \( T \) be the weight \( k \) Hecke algebra for \( \Gamma_0(N) \) and let \( m \subset T \) be a non-Eisenstein maximal ideal of residual characteristic \( l \). Denote the two dimensional representation attached to \( m \) by \( W \) and set \( \overline{\partial}_l = \partial_l/l\partial_l \).

THEOREM 7 (FAUTINGS). Suppose that \((l, N) = 1 \) and \( l > \max(3, k) \). Then \( H^1_{par}(Y_0(N) \times \overline{Q}_l, \overline{\partial}_l)[m] \) is two-dimensional over \( T/m \).

PROOF (SKETCH). By [Fa 1], the \( \text{Gal}(\overline{Q}_l/Q_l) \)-representation

\[
V = H^1_{par}(Y_0(N) \times \overline{Q}_l, \overline{\partial}_l)
\]

is crystalline. Let \( M \) be the \( F \)-crystal corresponding to the dual of \( V \). Then by imitating [Fa 2] one obtains on \( M \) a canonical Frobenius filtration

\[
M = F^0 \supset F^{k-1} \supset 0,
\]
with $F^{k-1} \approx S_k(\Gamma_0(N)) \otimes \overline{\mathbf{F}}$. As in [Ma], one proves that $F^0[m]$ is $W$-isotypical. Then multiplicity one for $S_k(\Gamma_0(N))$ implies that $W$ occurs in $F^0[m]$ only once.

IV. The Proof of Theorem 1. Let $T$ be the weight $k$ Hecke algebra for $\Gamma_0(pM)$ and suppose $m \subset T$ is the non-Eisenstein maximal ideal associated with $\rho$. Put $k = T/m$ and let $W$ be the representation space of $\rho$. Hence $\dim_k W = 2$.

**Theorem 8 (After Mazur).** Theorem 1 is true if $p \neq 1 \mod l$.

**Proof.** $W \subset H^1_{par}(Y_0(pM) \times \overline{\mathbf{Q}}, \overline{\mathbf{Q}})$ as a $T[\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})]$-module since $m$ is not Eisenstein. By Theorem 5, $W \subset H^1_{par}(Y_0(pM) \times \overline{\mathbf{F}}_p, \overline{\mathbf{Q}})$ as a $T[\text{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)]$-module since $\rho$ is unramified and $\Phi_p(X_0(pM))_l = 0$. If Theorem 1 is false, so that $W$ has no intersection with

$$H^1_{par}(Y_0(M) \times \overline{\mathbf{F}}_p, \overline{\mathbf{Q}}),$$

then by Theorem 6 $W \subset \hat{X}_p(X_0(pM))_l \otimes \mathbf{Z}/l\mathbf{Z}$. Frobp, wp, and Tp coincide here up to sign, so in particular Frobp is in T and acts on $W$ via its image in $T/mT = k$. Now $w_p^2 = (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})\gamma$ where $\gamma \in \Gamma$ and the scalar matrix $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$ acts on $\text{Symm}^{k-2}$ by $p^{k-2}$. Hence Frobp acts on $W \subset \hat{X}_p(X_0(pM))_l \otimes \mathbf{Z}/l\mathbf{Z}$ by multiplication by $\pm p^{(k-2)/2}$ and the determinant of its action is $p^{k-2}$. On the other hand the determinant of Frobp must be $\chi^{k-1}(\text{Frob} p) = p^{k-1}$ where $\chi$ is the $l$-cyclotomic character. So $p^{k-2} \equiv p^{k-1} \mod l$ and $p \equiv 1 \mod l$.

This same argument together with multiplicity 2 (Theorem 7) gives (compare [Ri]).

**Theorem 9.** Suppose $p \neq 1 \mod l$ and $X_l = \hat{X}_p(X_0(pM))_l$. Then $\dim_k X_l/lX_l \leq 1$.

We are supposing $W$ is unramified at $p$ and comes from $S_k(\Gamma_0(pM))^{p-\text{new}}$. By Corollary 4 there exists a prime $q > lM$ such that $q \equiv 1 \mod l$, and such that $W$ comes from $S_k(\Gamma_0(pqM))^{p-\text{new},d-\text{new}}$. Hence

$$W \subseteq H^1(V_B(M) \times \overline{\mathbf{Q}}, \overline{\mathbf{Q}}).$$

Now apply Theorem 5: if $W$ has a nonzero projection to $\Phi_p(V_B(M))_l$ then we are done since $\Phi_p(V_B(M))_l$ is a module of fusion between $p$-old and $p$-new forms by Theorem 3. By Theorem 6 we can suppose $W$ lives in $\hat{X}_p(V_B(M))_l$. But now consider the Ribet Exact Sequence (Theorem 2):

$$0 \to \hat{X}_q(X_0(qM))^2 \to \hat{X}_q(X_0(pqM))_l \to \hat{X}_p(V_B(M))_l \to 0.$$

If $m$ is in the support of $\hat{X}_q(X_0(qM))^2$, then we are done since by Theorem 8 we can remove the $q$ from the level $qM$. So assume that $m$ is not in the support of $\hat{X}_q(X_0(qM))^2$ and we obtain:

$$\text{LHS} \overset{\text{def}}{=} \hat{X}_q(X_0(pqM))_l/m\hat{X}_q(X_0(pqM))_l \approx \hat{X}_p(V_B(M))_l/m\hat{X}_p(V_B(M))_l \overset{\text{def}}{=} \text{RHS}.$$

$W$ living in $\hat{X}_p(V_B(M))_l$ shows that $\dim_k(\text{RHS}) \geq 2$. But Theorem 9 shows that $\dim_k(\text{LHS}) \leq 1$, a contradiction.
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