Cartan subalgebra need not give rise to a Laplace operator on the supergroup. The author shows, however, that under suitable hypotheses, satisfied for $U(p, q)$ and $C(m, n)$, one can recover the classical result by considering instead *rational* Weyl invariant functions on the Cartan and the *field of fractions* of the algebra of Laplace-Casimir operators. Unforturnately, this interesting idea is tossed of rather lightly, leaving its meaning unclear (at least to the reviewer).

REFERENCES

[**B**] M. Batchelor, *The structure of supermanifolds*, Trans. Amer. Math. Soc. **253** (1979), 329–338.

[**Ga**] K. Gawedzki, *Supersymmetries—mathematics of supergeometry*, Ann. Inst. H. Poincaré Sect. A (N.S.) **27** (1977), 335–366.

[**Gr**] P. Green, *On holomorphic graded manifolds*, Proc. Amer. Math. Soc. **85** (1982), 587–590.

[**K**] B. Kostant, *Graded manifolds, graded Lie theory and prequantization*, Differential Geometric Methods in Mathematical Physics (Proc. Sympos. Univ. Bonn, Bonn, 1975), Lecture Notes in Math., vol. 570, Springer-Verlag, Berlin, Heidelberg and New York, 1977, pp. 177–306.

[**M**] Yu. I. Manin, *Gauge field theory*, Grundlehren Math. Wiss. vol. 289, Springer-Verlag, Berlin, Heidelberg, and New York,

MITCHELL J. ROTHSTEIN
SUNY AT STONY BROOK

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 1, July 1989

0273-0979/89 $1.00 + $.25 per page

Gauge field theory and complex geometry, by Yuri I. Manin. Translated from the Russian by N. Koblitz and J. R. King, Springer-Verlag, Berlin, Heidelberg, 1988, x + 295 pp., $80.00. ISBN 0387-18275-6

Few areas of twentieth century scientific thought have provoked more puzzlement, outrage, and disbelief among the general populace than that part of modern physics which asserts that the true geometry of the natural world is profoundly different from the "common sense" geometry canonized by Euclid. The definitive revolution in this area was, of course, that wrought by Einstein, Minkowski, and their contemporaries, who discovered a new geometry, not of space but rather of space-time. This change of perspective had, in retrospect, been waiting to happen ever since Maxwell wrote down his field equations for electromagnetism; it was not new physics, but rather the casting of the symmetries of the old physics in a geometrical guise, that brought the new geometry into existence.

In the last decades, a number of new geometric ideas have entered the arena of theoretical physics, often with lasting repercussions for mathematics. The present book deals with three families of such ideas: those of nonabelian gauge-field theory, of twistor theory, and of supersymmetry. The main thrust of the work centers on an exegesis of a paper in which Ed

Witten [14] discovered a remarkable interplay between these three families of ideas, but the reader will encounter along the way a number of interesting digressions which lead to results untreated elsewhere in the literature.

Gauge theory in the present context means the study of the Yang-Mills equations. On a vector bundle E over a pseudo-Riemannian manifold M, a connection ∇ is said to satisfy the Yang-Mills equations if its curvature 2-form F is co-closed:

$$^*d_\nabla {}^*F = 0$$

where * denotes the Hodge-star operator and d_∇ is the lift via ∇ of the de Rham exterior derivative to bundle-valued forms. (Equivalently, we demand that $\nabla^a F_{ab} = 0$.) If M is Minkowski space and $E \to M$ is a rank one bundle (i.e. a line bundle), F is a 2-form in the elementary sense, and its six components are simply required to satisfy the equations Maxwell wrote down to govern the three components of the electric field and the three components of the magnetic field. If, on the other hand, M is a compact Riemannian manifold, but if $E \to M$ is again rank one, the Yang-Mills equations just demand that the curvature of E be the unique harmonic 2-form which, by Hodge theory, represents the Chern class of the line bundle. The Yang-Mills equations thus may be thought of as natural generalizations of both Maxwell's equations and the Hodge equations for a harmonic form. Unlike these equations, however, the Yang-Mills equations become nonlinear when the fibers of our vector bundle have higher dimension. These equations (or, strictly speaking, the quantum theory of the action from which they spring) are now widely believed to govern the strong force which binds together the nuclei of atoms, and a slight variant is used to account for the weak force which regulates the beta decay of neutrons. Through the remarkable work of Simon Donaldson [4], the Yang-Mills equations have, in addition, become the major tool in four-dimensional differential topology.

Twistor theory, the second family of geometric ideas featured here, originated out of a series of insights of Roger Penrose [12] concerning relations between the conformal geometry of Minkowski space, complex analysis, and the solutions of certain conformally invariant differential equations such as Maxwell's equations. Consider the 2-sphere of all light-rays through a point of space-time; we may think of this as the field of vision of an ideal observer capable of looking in all directions at once. One might ask how the rest-frame of this observer influences the geometry of his field of vision. The answer is that its *conformal* geometry is independent of rest-frame, so that the picture of the world seen by two colliding observers at their moment of collision would only differ by a Möbius transformation of the Riemann sphere. In short, the 2-sphere of light-rays through a point in space-time has the structure a complex 1-manifold. In flat space-time, these complex structures fit neatly together, to the following beautiful effect: the set of all light-rays in Minkowski space has the structure of a 5-dimensional real hypersurface in a complex 3-manifold; namely, it can be identified with a real hyperquadric in complex projective 3-space minus a projective line. This $\mathbf{CP}_3$ is known as (projective) twistor space. It turns out that holomorphic objects (bundles, cohomology classes, etc.) on this

$\mathbf{CP}_3$ (or suitable regions thereof) correspond to solutions of conformally invariant PDE's on Minkowski space via the so-called *Penrose transform*, examples of which were originally discovered by Penrose in the guise of contour integral formulae producing solutions of the massless Dirac equations of all helicities. A particularly beautiful version of this correspondence was discovered when Richard Ward [**13**] realized that holomorphic line bundles on twistor space correspond to solutions of Maxwell's equations on Minkowski space and that, more generally, holomorphic vector bundles on twistor space give rise to self-dual solutions of the Yang-Mills equations on space-time. Here a connection on a bundle over Minkowski space is called self-dual if its curvature satisfies $F = i * F$, where again, $*$ is the Hodge star-operator. Such a connection cannot have a covariantly constant fiber-wise inner product, and so is excluded as a physically admissible classical solution of the Yang-Mills equations, but such solutions are nonetheless considered to have an important influence on the quantum version of the theory, where they allegedly give rise to tunneling behavior. Essentially the same Penrose transform may instead be used to interpret holomorphic objects on twistor space as solutions of conformally invariant elliptic PDE's on Euclidean 4-space, this picture being related to the Minkowskian one by analytic continuation, or "Wick rotation;" the self-duality equations on Euclidean space became the more acceptable $F = *F$, and it is suddenly possible to have solutions with invariant inner products. These Euclidean solutions are the so-called *instantons*, which were classified by Atiyah-Hitchin-Drinfeld-Manin [**1**] under the assumption that the solution is defined on all of Euclidean 4-space and has L^2 curvature, the latter being equivalent to saying that the solution extends to S^4. The key tool in this classification is the twistor correspondence, which reduces the problem to one of classifying algebraic vector bundles on $\mathbf{CP}_3$ subject to certain extra conditions, the latter question having already been studied extensively by algebraic geometers.

The unsatisfactory aspect of the above state of affairs is that the most physically interesting solutions of the Yang-Mills equations are *not* the self-dual ones, but rather those which have compact holonomy on Minkowski space (or perhaps some other Lorentz-signature space-time). Thus the search began for a twistor-like correspondence for the general solutions of the Yang-Mills equations, with the self-duality condition dropped. Results of this kind were eventually found by Isenberg, Yasskin and Green [**6**] and by Witten [**14**]. In this correspondence, twistor space $\mathbf{CP}_3$ is replaced by a certain nonreduced complex projective variety, namely the third infinitesimal neighborhood of

$$A = \left\{ ([Z^0, Z^1, Z^2, Z^3], [W_0, W_1, W_2, W_3]) \in \mathbf{CP}_3 \times \mathbf{CP}_3 \,\middle|\, \sum_{j=0}^{3} Z^j W_j = 0 \right\}$$

in $\mathbf{CP}_3 \times \mathbf{CP}_3$. Geometrically, A represents the space of null lines in complex Minkowski space.

While Isenberg *et al.* gave a direct, albeit complicated, proof of this correspondence, Witten offered a brief, dazzling argument involving supersymmetry and complex supermanifolds. For the present purposes, the appropriate definition of a supermanifold is that of Leites and Berezin **[3]** (arrived at somewhat later by Kostant **[7]**), which provides a slick way of making sense out of the physicist's vague notion of "a manifold, some of whose coordinates anticommute." In the Leites-Berezin formalism, this simply means that one considers an enlargement of the sheaf of functions of some manifold by the adjoining of some anticommuting objects; anyone who has ever manipulated differential forms has, from this point of view, played with a supermanifold, albeit a rather dull one. Manin devotes a full half of the present work to the systematic development of the theory of complex supermanifolds, and by so doing simultaneously fills a gaping hole in the expository literature and provides us with an important piece of original research. Unlike some attempts to develop the mathematical idea of a supermanifold, the present work not only provides insights into the foundations of the subject but goes on to tell us what all this has to do with the physical motivation, with nice treatments of the super-Yang-Mills equations and $N = 1$ supergravity. In particular, an account is given of Witten's supersymmetric twistor correspondence for the super-Yang-Mills equations, as well as its relation to the previously mentioned twistor correspondence of Isenberg *et al.*

Because the original Russian version was written the better part of a decade ago, many recent developments, answering questions raised in the text, are necessarily left untreated. In particular, the work of Uhlenbeck, Taubes, and Donaldson has shed unexpected light onto the structure of the moduli spaces of self-dual Yang-Mills fields on compact Riemannian 4-manifolds. Moreover, we now have a good understanding of the correct curved space-time analog of the twistor correspondence for non-self-dual Yang-Mills fields, and deep connections have turned up between this correspondence, conformal gravity and supergravity (cf. **[2, 8, 9, 15]**). As for complex supermanifolds, the subject has been given renewed interest by the advent of superstring theory **[15, 16]** and the structure of the moduli space of super-Riemann surfaces **[10, 11]** has therefore become a popular topic. The important point is that this book provides a large amount of background for current research in across a spectrum of fields. While many readers will no doubt find much of the book heavy going, there are many rewards to be reaped here in exchange for the effort expended. The translators have thus done us a notable good turn by producing this serviceable English version.

References

1. M. Atiyah, N. Hitchin, V. Drinfeld, and Y. Manin, *Construction of instantons*, Phys. Lett. **65A** (1978), 185–187.

2. R. Baston and L. Mason, *Conformal gravity, the Einstein equations, and spaces of complex null geodesics*, Class Quantum Grav. **4** (1987), 815–826.

3. F. Berezin and D. Leites, *Supermanifolds*, Soviet Math. Dokl. **16** (1975), 1218–1221.

4. S. Donaldson, *The geometry of* 4-*manifolds*, Proc. Internat. Congr. Math., Berkeley, 1986, pp. 43–54; Amer. Math. Soc., Providence, R.I., 1987.