ON SOLVABLE SUBGROUPS OF THE SYMMETRIC GROUP

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1. Introduction. In this note we give exact values of certain invariants of the symmetric group $S_n$ of degree $n$.

Let $n$ be a positive integer, $p$ a prime, $\sigma(G)$ the derived length and $\nu(G)$ the nilpotent length of a solvable group $G$. Let $\text{SOLV}(n)$ denote the set of all solvable subgroups of $S_n$ and put

$$\text{SOLV}(n,p') = \{G \in \text{SOLV}(n) | p \nmid |G|\},$$

$$\sigma(n) = \max\{\sigma(G) | G \in \text{SOLV}(n)\},$$

$$\nu(n) = \max\{\nu(G) | G \in \text{SOLV}(n)\}.$$}

Similarly one defines $\sigma(n,p')$ and $\nu(n,p')$.

Let $N$ be the set of all nonnegative integers. For $t \in N$ we put $s(t) = \min\{m \in N | \sigma(m) = t\}$ and $n(t) = \min\{m \in N | \nu(m) = t\}$. For a partial ordered set $L$ we denote by $\mu L$ the set of all maximal elements in $L$. We put $\Sigma(t) = \{G \in \mu \text{SOLV}(s(t)) | \sigma(G) = t\}$ and $\Sigma(t,p') = \{G \in \mu \text{SOLV}(s(t,p'),p') | \sigma(G) = t\}$. Similarly one defines $N(t)$ and $N(t,p')$.

We define the structure of all elements of the sets $\Sigma(t)$, $\Sigma(t,p')$, $N(t)$ and $N(t,p')$.

We assume that, as permutations groups, $S_m$ has degree $m$, $\text{AGL}(2,3)$ has degree 9, the cyclic group $C(p)$ of order $p$ has degree $p$, the groups $\text{AGL}(1,p)$ and $\frac{1}{2} \text{AGL}(1,p)$ (=the subgroup of index 2 in $\text{AGL}(1,p)$) have degree $p$.

We say that a group $W$ is of type $\{B_1^{k_1}, \ldots, B_s^{k_s}\}$ if $W$ a wreath product of $k_1$ copies of the permutation group $B_1$, $k_2$ copies of the permutation group $B_2$ and so on (the order of the factors is arbitrary).

2. Main results.

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THEOREM 1. Let \( G_t \in \Sigma(t) \). If \( t < 4 \), then \( G_t = S_{t+1} \). If \( t = 4 \), then \( G_t \) is of type \( \{S_2, S_4\} \). Suppose now that \( t > 4 \).

(a) \( G_{5k} \) is of type \( \{\text{AGL}(2, 3)^k\} \) (so \( s(5k) = 9^k \)).
(b) \( G_{5k+1} \) is of type \( \{S_4^2, \text{AGL}(2, 3)^{k-1}\} \).
(c) \( G_{5k+2} \) is of type \( \{S_3, \text{AGL}(2, 3)^k\} \).
(d) \( G_{5k+3} \) is of type \( \{S_4, \text{AGL}(2, 3)^k\} \).
(e) \( G_{5k+4} \) is of type \( \{S_4^3, \text{AGL}(2, 3)^{k-1}\} \) and \( s(5k + 4) = 4^3 \cdot 9^{k-1} \).

If the function \( s \) is known, one can restore \( \sigma \).

THEOREM 2. Let \( G_t \in \Sigma(t, 2') \). Then

(a) \( G_{2k} \) is of type \( \{\frac{1}{2} \text{AGL}(1, 7)^k\} \) and \( s(2k, 2') = 7^k \).
(b) \( G_{2k+1} \) is of type \( \{C(3), \frac{1}{2} \text{AGL}(1, 7)^k\} \) and \( s(2k + 1, 2') = 3 \cdot 7^k \).

THEOREM 3. If \( G_t \in \Sigma(t, 3') \), then \( G_t \in \text{Syl}_2(S_{2t}) \).

THEOREM 4. If \( p > 3 \), then \( \Sigma(t, p') = \Sigma(t) \).

THEOREM 5. Let \( G_t \in N(t) \). Then \( G_1 = S_2 \). Suppose that \( t > 1 \).

(a) \( G_{4k} \) is of type \( \{\text{AGL}(2, 3)^a, S_3^{2(k-a)}\} \), \( a \leq k \).
(b) \( G_{4k+1} = S_4 \wr H \) where \( H \) is of type \( \{\text{AGL}(2, 3)^a, S_3^{2(k-a)-1}\} \), \( a < k \).
(c) \( G_{4k+2} \) is of type \( \{\text{AGL}(2, 3)^a, S_3^{2(k-a)+1}\} \), \( a \leq k \).
(d) \( G_{4k+3} = S_4 \wr G_{4k} \) and \( s(4k + 3) = 4 \cdot 3^{2k} \).

THEOREM 6. Let \( G_t \in N(t, 2') \). Then

(a) \( G_{2k} \in \Sigma(2k, 2') \) and \( N(2k, 2') = \Sigma(2k, 2') \).
(b) \( G_{2k+1} = C(3) \wr G_{2k} \in \Sigma(2k + 1, 2') \) (but \( N(2k + 1, 2') \neq \Sigma(2k + 1, 2') \)).

THEOREM 7. Let \( G_t \in N(t, 3') \). Then

(a) \( G_{2k} \) is of type \( \{\text{AGL}(1, 5)^k\} \) and \( s(2k, 3') = 5^k \).
(b) \( G_{2k+1} = C(2) \wr G_{2k}, \ s(2k + 1, 3') = 2 \cdot 5^k \).

THEOREM 8. If \( p > 3 \), then \( N(t, p') = N(t) \).

If \( G = p_1^{n_1} \cdots p_s^{m_s} \), then \( \lambda(G) = m_1 + \cdots + m_s \). If \( G \) is solvable, its composition length \( c(G) \) is equal to \( \lambda(G) \). We put

\[ c(n) = \max\{c(G) | G \in \text{SOLV}(n)\}. \]

Similarly one defines \( c(n, p') \).

THEOREM 9. Let \( G \in \text{SOLV}(n) \) be transitive and \( c(G) = c(n) \). Then

(a) \( n = 4^k \), \( G \) is of type \( \{S_4^k\} \).
(b) \( n = 2 \cdot 4^k \), \( G = H \wr S_2 \) where \( H \) is from (a).
(c) \( n = 3 \cdot 4^k \), \( G = H \wr S_3 \) where \( H \) is from (a).
(d) \( n = 6 \cdot 4^k \), \( G = H \wr F \) where \( H \) is from (a), \( F \) is of type \( \{S_2, S_3\} \).
THEOREM 10. Let $G \in \text{SOLV}(n, 2')$ be transitive and $c(G) = c(n, 2')$. Then
(a) $n = 3^k$, $G \in \text{Syl}_3(S_n)$.
(b) $n = 5 \cdot 3^k$, $G = H \text{wr} \text{C}(5)$ where $H$ is from (a).
(c) $n = 7 \cdot 3^k$, $G = H \text{wr} \frac{1}{2} \text{AGL}(1, 7)$ where $H$ is from (a).

THEOREM 11. Let $G \in \text{SOLV}(n, 3')$ be transitive and $c(G) = c(n, 3')$. Then
(a) $n = 2^k$, $G \in \text{Syl}_2(S_n)$.
(b) $n = 5 \cdot 2^k$, $G = H \text{wr} \text{AGL}(1, 5)$ where $H$ is from (a).

THEOREM 12. If $p > 3$, then $n$ is the same as in Theorem 9 and $c(n, p') = c(n)$ for any transitive $G \in \text{SOLV}(n, p')$ with $c(G) = c(n, p')$; $G$ is the group from Theorem 9.

We put
$$o(n) = \max\{|G| | G \in \text{SOLV}(n)\}$$
and
$$o(n, p') = \max\{|G| | G \in \text{SOLV}(n, p')\}.$$
THEOREM 19. Let $N$ be a nilpotent subgroup of maximal $p'$-order in $S_n$, $p > 2$. If $p = 3$, then $N \in \text{Syl}_2(S_n)$. If $p > 3$, then $N$ be the group from Theorem 17.