Finiteness and Vanishing Theorems for Complete Open Riemannian Manifolds

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Let $M^n$ denote an $n$-dimensional complete open Riemannian manifold. In [AG] Abresch and Gromoll introduced a new concept of "diameter growth." Roughly speaking, one would like to measure the essential diameter of ends at distance $r$ from a fixed point $p \in M^n$. They showed that $M^n$ is homotopy equivalent to the interior of a compact manifold with boundary if $M^n$ has nonnegative Ricci curvature and diameter growth of order $o(r^{1/n})$, provided the sectional curvature is bounded from below. It is well known that any complete open manifold with nonnegative sectional curvature has finite topological type. This is a weak version of the Soul Theorem of Cheeger-Gromoll [CG]. Examples of Sha and Yang show that this kind of finiteness result does not hold for complete open manifolds with nonnegative Ricci curvature in general (see [SY1, SY2]), and additional assumptions are therefore required.

We will use a concept of the essential diameter of ends slightly stronger than that of [AG]: For any $r > 0$, let $B(p,r)$ denote the geodesic ball of radius $r$ around $p$. Let $C(p,r)$ denote the union of all unbounded connected components of $M^n \setminus B(p,r)$. For $r_2 > r_1 > 0$, set $C(p;r_1, r_2) = C(p,r_1) \cap B(p,r_2)$. Let $1 > \alpha > \beta > 0$ be fixed numbers. For any connected component $\Sigma$ of $C(p;\alpha r, \beta r)$, and any two points $x, y \in \Sigma \cap \partial B(p,r)$, consider the distance $d_r(x,y) = \inf \text{Length}(\phi)$ between $x$ and $y$ in $C(p,\beta r)$, where the infimum is taken over all smooth curves $\phi \subset C(p,\beta r)$ from $x$ to $y$. Set $\text{diam}(\Sigma \cap \partial B(p,r), C(p,\beta r)) = \sup d_r(x,y)$, where $x,y \in \Sigma \cap \partial B(p,r)$. Then the diameter of ends at distance $r$ from $p$ is defined by

$$\text{diam}(p,r) = \sup \text{diam}(\Sigma \cap \partial B(p,r), C(p,\beta r)),$$

where the supremum is taken over all connected components $\Sigma$ of $C(p;\alpha r, \frac{1}{\alpha} r)$. The diameter defined here is not smaller than that defined by Abresch and Gromoll. Our definition will be essential in Lemma 3 and its applications.

The purpose of this note is to announce the following results.

**Theorem A.** Let $M$ be a complete open Riemannian manifold with sectional curvature $K_M \geq -K^2$ for some constant $K > 0$. Assume that for some base point $p \in M$,

$$\limsup_{r \to +\infty} \text{diam}(p,r) < \frac{\ln 2}{K}.$$

Received by the editors February 6, 1989 and, in revised form, May 25, 1989. 1980 Mathematics Subject Classification (1985 Revision). Primary 53C20.
Then $M$ is homotopy equivalent to the interior of a compact manifold with boundary.

**Theorem B.** Let $M^n$ be an $n$-dimensional complete open Riemannian manifold. Suppose that the sectional curvature $K_M \geq -K^2$ for some constant $K > 0$. Assume that for some $2 \leq k \leq n - 1$, $M^n$ has nonnegative $k$th-Ricci curvature and that for some $p \in M^n$,

$$
\limsup_{r \to +\infty} \frac{\text{diam}(p, r)}{r^{\frac{2k}{k+1}}} < \left[ \frac{2(k+1)}{k} \left( \frac{(k-1) \ln 2}{2kK} \right)^{k/(k+1)} \right].
$$

Then $M^n$ is homotopy equivalent to the interior of a compact manifold with boundary.

**Theorem C.** Let $M^n$ be an $n$-dimensional complete open Riemannian manifold. Assume that for some $1 \leq k \leq n - 1$, $M^n$ has positive $k$th-Ricci curvature everywhere and that for some $p \in M^n$, $M^n$ has diameter growth of order $o(r)$, i.e.

$$
\limsup_{r \to +\infty} \frac{\text{diam}(p, r)}{r} = 0.
$$

Then $M^n$ has the homotopy type of a CW-complex with cells of dimensions $\leq k - 1$.

The precise condition that $M^n$ have nonnegative (positive) $k$th-Ricci curvature at some point $x \in M^n$ is that for all $v$ in the span of any orthonormal set $\{e_1, \ldots, e_{k+1}\}$ in $T_x M^n$,

$$
\sum_{i=1}^{k+1} \langle R(e_i, v) v, e_i \rangle \geq 0 \ (> 0),
$$

where $R(x, y) z$ denotes the curvature tensor of $M^n$ (cf. also [H] for the definition of $k$th-Ricci curvature).

**Remark 1.** (1) In Theorem A the upper bound $\ln 2/K$ must depend on $K$. Otherwise, the connected sum of infinitely many copies of $S^2 \times S^2$ (see [AG]) provides an easy counterexample.

(2) Theorem B generalizes the Abresch-Gromoll Theorem [AG].

(3) The condition in Theorem C can be weakened to that $M^n$ has nonnegative $k$th-Ricci curvature everywhere and positive $k$th-Ricci curvature outside a compact subset of $M^n$ (see Lemma 5).

(4) The same argument as in [AG] shows that any complete open Riemannian manifold with nonnegative Ricci curvature must have diameter growth of order $o(r)$. We do not know whether the condition in Theorem C on diameter growth is necessary. Examples in [SY1, SY2, We and GM] have diameter growth of order at most $o(r)$.

It is a pleasure to thank D. Gromoll for some valuable suggestions. I would also like to thank G. Gong, A. Phillips and G. Wei for helpful discussions.

**Outline of Proofs.** Throughout this part we assume that $M^n$ denotes a complete open Riemannian manifold of dimension $n$ and $p$ is a point of
fixed during the discussion. For arbitrary \( t \geq 0 \), let \( R_t(p) = \{ \gamma(t); \gamma \) is a ray emanating from \( p \} \), which is a closed subset of the distance sphere \( S(p, t) \). Set \( B^t_p(x) = t - d(x, R_t(p)) \) for any \( x \in \mathcal{M}^n \). It is easy to see that \( B^t_p(x) \) is increasing in \( t \) and \( |B^t_p(x)| \leq d(p, x) \) for any \( x \in \mathcal{M}^n \). The generalized Busemann function \( B_p \) is defined as \( B_p(x) = \lim_{t \to +\infty} B^t_p(x) \), which is a Lipschitz function with Lipschitz constant 1. The excess function \( E_p \) is defined as \( E_p(x) = d(p, R_t(p)) - B_p(x) \). We will introduce a new function \( L_p \) which plays an essential role in the study of the generalized Busemann function \( B_p \). Set \( L_p(x) = d(x, R_t(p)) \), where \( t = d(p, x) \). Since \( B^t_p(x) \) is increasing in \( t \), it is easy to see that \( E_p(x) \leq L_p(x) \) and \( d(p, x) - L_p(x) \leq B_p(x) \) for all \( x \in \mathcal{M}^n \). A more detailed discussion for generalized Busemann functions has been given by H. Wu [W1]. For the purpose of this note, we need the following

**Lemma 1.** For any \( q \in \mathcal{M}^n \), there exists a ray \( \sigma_q(t) \) emanating from \( q \) such that for all \( t \geq 0 \), the function \( B^t_q(x) \) defined by \( B_p(q) + t - d(x, \sigma_q(t)) \) supports \( B_p(x) \) at \( q \), namely \( B^t_q(x) \leq B_p(x) \) for all \( x \in \mathcal{M}^n \) and \( B^t_q(q) = B_p(q) \).

**Lemma 2.** Suppose that \( \mathcal{M}^n \) has sectional curvature \( K_M \geq -K^2 \) for some \( K > 0 \), then for any critical point \( q \) with respect to \( p \),

\[
E_p(q) \geq \frac{1}{K} \left( e^{Kd(p,q)} \right).
\]

Notice that \( E_p(x) \leq L_p(x) \) for all \( x \in \mathcal{M}^n \). Thus if \( \limsup_{d(p, x) \to +\infty} L_p(x) < \frac{\ln 2}{K}, \) Lemma 2 shows that outside a compact subset there is no critical point with respect to \( p \), Theorem A follows from this argument and the following

**Lemma 3.** Suppose that \( \mathcal{M}^n \) has diameter growth of order \( o(r) \). Then there exists an \( R > 0 \) such that for any \( x \in \mathcal{M}^n \setminus B(p, R) \),

\[
(1) \quad L_p(x) \leq \text{diam}(p, d(p, x)),
\]

and the Busemann function \( B_p \) is proper.

Notice that \( d(p, x) - L_p(x) \leq B_p(x) \) for all \( x \in \mathcal{M}^n \). It is clear that (1) implies that \( g(x) \equiv d(p, x) - L_p(x) \) is proper, and so is \( B_p(x) \).

One can obtain a better estimate for \( E_p \leq L_p \) in terms of \( L_p \) if \( \mathcal{M}^n \) has nonnegative \( k \)th-Ricci curvature.

**Lemma 4.** Suppose that \( \mathcal{M}^n \) has nonnegative \( k \)th-Ricci curvature for some \( 2 \leq k \leq n - 1 \), then for all \( x \in \mathcal{M}^n \) with \( L_p(x) < d(p, x) \),

\[
(2) \quad E_p(x) \leq \frac{2k}{k-1} \left[ \frac{k}{2(k+1)} \times \frac{L_p(x)^{k+1}}{d(p, x) - L_p(x)} \right]^{1/k}.
\]

The proof of Lemma 4 depends on Lemma 1 and the maximum principle. Theorem B therefore follows from Lemmas 2, 3, and 4. For the proof of Theorem C, we need Lemma 3 and the following
Lemma 5. Suppose that for some $1 \leq k \leq n - 1$, $M^n$ has nonnegative $k$th-Ricci curvature everywhere and positive $k$th-Ricci curvature outside a compact subset. If the Busemann function $B_p$ is proper, then there exists a $C^2$ function $\chi(t)$ such that $\chi \circ B_p$ is proper and strictly $k$-convex. Therefore $M^n$ has the homotopy type of a CW-complex with cells of dimensions $\leq k - 1$.

Compare [W2] for a definition of $k$-convexity. It seems to be crucial that the Busemann function $B_p$ is proper. The first assertion in Lemma 5 follows from Lemma 1. If we assume that $\chi \circ B_p$ is proper and strictly $k$-convex, then the last assertion in Lemma 5 follows from Wu's Smoothing Theorem [W2] and the standard Morse Theory [M]. This proves Theorem C.

Remark 2. An observation of Cheeger-Gromoll ([CG], sharpened in [GW]) is that if $M^n$ has nonnegative sectional curvature outside a compact subset, then $M^n$ has finite topological type and $B_p$ is a proper function. If an addition, $M^n$ has nonnegative $k$th-Ricci curvature everywhere and positive $k$th-Ricci curvature outside a compact subset, then $M^n$ has the homotopy type of a CW-complex with finitely many cells of dimensions $\leq k - 1$ (cf. [W2]).

References

[SY2] ———, Positive Ricci curvature on the connected sums of $S^n \times S^m$, Preprint.

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