HEAT CONDUCTION FOR RIEMANIAN FOLIATIONS
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A foliation $\mathcal{F}$ on a manifold $M$ is a partition of $M$ into submanifolds, the leaves of $M$, which locally looks like a family of parallel subspaces in Euclidean space. This gives rise to two types of geometries, namely tangential and transversal.

More specifically let $g_M$ be a Riemannian metric on $M$. It induces on each leaf a Riemannian metric, and hence a corresponding leafwise Laplacian $\Delta_0$. The action of the corresponding semigroup $e^{-t\Delta_0}$ on the bigraded de Rham complex $\Omega_M$ is studied in [AT], and leads to a tangential or leafwise Hodge decomposition theorem.

A foliation $\mathcal{F}$ is Riemannian [R], if the induced Riemannian metric $g_Q$ on the normal bundle $Q = TM/L$, $L$ the tangent bundle of $\mathcal{F}$, is holonomy invariant, i.e. $\theta(X)g_Q = 0$ for all vector fields $X$ tangent to $\mathcal{F}$. This gives rise to a transversal Riemannian geometry, which can heuristically be thought of as the Riemannian geometry of the (singular) space of leaves $B = M/\mathcal{F}$. The complex $\Omega_B(\mathcal{F}) \subset \Omega_M$ of forms $\omega$ satisfying $i(X)\omega = 0$ (interior product) and $\theta(X)\omega = 0$ (Lie derivative) for all $X \in TL$ is the complex of basic differential forms of $\mathcal{F}$, and heuristically plays the role of the de Rham complex of the leaf space $B$. The transversal Riemannian metric $g_Q$ gives rise to a transversal or basic Laplacian $\Delta_B: \Omega_B(\mathcal{F}) \rightarrow \Omega_B(\mathcal{F})$. The main point of this announcement is to construct and study the corresponding semigroup $e^{-t\Delta_B}$ acting on $\Omega_B(\mathcal{F})$, and to examine its limit behavior for $t \rightarrow \infty$. This yields in particular a new proof of the Hodge decomposition theorem in $\Omega_B(\mathcal{F})$.

The Laplacian $\Delta_B = d_B\delta_B + \delta_B d_B$ is formally constructed from the transversal Riemannian geometry in the normal bundle in the usual fashion. Since the basic differential forms do not constitute all sections of a vector bundle, the usual elliptic theory does not apply directly. A technical device to handle this situation is to extend $\Delta_B: \Omega_B(\mathcal{F}) \rightarrow \Omega_B(\mathcal{F})$ to a genuine elliptic operator $\hat{\Delta}: \Omega(M) \rightarrow \Omega(M)$. An explicit construction of such an extension was given in [KT]. This involves the assumption that the mean curvature is constant along the leaves. This hypothesis enters in the explicit calculation of the formal adjoint $\delta_B$ of $d_B$, and hence also of $\Delta_B$. It is further used in the construction of the extension $\hat{\Delta}$. Formally this hypothesis is expressed by dualizing the usual mean curvature vector field to a 1-form $\kappa$ (vanishing along the leaves of $\mathcal{F}$), and requiring it to be an element of $\Omega_B^1(\mathcal{F})$. The reason for this assumption is that it makes the

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construction of \( \tilde{\Delta} \) as explained above possible. It is not known if the results stated below hold in a more general context. However, the mean curvature hypothesis holds for several interesting classes of foliations, including Riemannian submersions (for appropriate metrics), E. Cartan's isoparametric families of surfaces [C], and foliations by the orbits of isometric Lie group actions (\( B \) is then an orbifold). A typical example of the latter kind is the flow defined by a nonsingular Killing field. The mean curvature form is invariant under the flow, hence constant along the orbits.

For an initial \( r \)-form \( \alpha_0 \in \Omega^r_B(\mathcal{F}) \), \( 0 \leq r \leq q = \text{codim}(\mathcal{F}) \), consider then the heat equation

\[
\frac{\partial}{\partial t} \alpha(x, t) = -\Delta_B \alpha(x, t), \quad \lim_{t \to 0} \alpha(x, t) = \alpha_0(x).
\]

In the situation described above, the main result is then as follows.

**Theorem.** Let \( \mathcal{F} \) be transversally oriented Riemannian foliation on a closed oriented manifold \((M, g_M)\). Assume \( g_M \) to be a bundle-like metric with \( \kappa \in \Omega^1_B(\mathcal{F}) \). Then the following holds.

(i) There exists a unique solution \( \alpha \) of (1), given in terms of the fundamental solution \( e_B^r(x, y, t) \) of the basic heat operator \( \partial / \partial t + \Delta_B \) by

\[
\alpha(x, t) = \int_M e_B^r(x, y, t) \wedge * \alpha_0(y).
\]

(ii) Denote \( \alpha(x, t) = [P_B(t)\alpha_0](x) \). Then there exists a uniform limit

\[
\lim_{t \to \infty} P_B(t)\alpha_0 = H_B\alpha_0 \in \Omega^r_B,
\]

and \( H_B\alpha_0 \) is \( \Delta_B \)-harmonic.

(iii) The form

\[
G_B\alpha_0 = \int_0^\infty (P_B(t)\alpha_0 - H_B\alpha_0) \, dt
\]

is well defined, and gives an operator \( G_B : \Omega^r_B \to \Omega^r_B \) satisfying

\[
\alpha_0 = \Delta_B G_B\alpha_0 + H_B\alpha_0.
\]

The finite-dimensionality of the space of basic harmonic \( r \)-forms \( \mathcal{H}^r_B = \ker \Delta_B \) is a consequence of the method of proof. The identity in (iii) implies in usual fashion the orthogonal Hodge decomposition

\[
\Omega^r_B \cong \text{im } d_B \oplus \text{im } \delta_B \oplus \mathcal{H}^r_B
\]

and the isomorphism \( H_B^r \cong \mathcal{H}^r_B \) (see [EH, and KT] for proofs of this result).

For the case of a point foliation, this is the approach to the classical Hodge decomposition pioneered by Milgram and Rosenbloom [MR].

A typical example is a Riemannian foliation \( \mathcal{F} \) transverse to the fibers of a flat bundle \( \tilde{X} \times_{\Gamma} F \), defined by an isometric action \( h : \pi_1(X) = \Gamma \to \text{Iso}(F) \) of \( \Gamma \) on the Riemannian manifold \( F \). In this case \( \Omega_B(\mathcal{F}) \cong \Omega(F)^\Gamma \), and the heat flow discussed is induced by the \( \Gamma \)-equivariant heat flow of the Laplacian on \( F \).

The novel technical aspect in the present context is the use of the elliptic extension \( \tilde{\Delta} \). For the case of basic functions (forms of degree 0)
treated in [EK] this difficulty does not arise, since the ordinary Laplacian on $M$ already preserves basic functions. The intuitive idea in part (i) of the theorem is to think of the flow of the basic heat operator $\partial/\partial t + \Delta_B$ as arising from the flow of the heat operator $\partial/\partial t + \tilde{\Delta}$ by restriction. The crucial property is then that an initial basic $r$-form $\alpha_0$ remains basic under this heat flow. This invariance property constitutes a parabolic version of Hadamard’s descent method for linear hyperbolic equations [H]. It is natural to expect that this heat equation approach will be useful for the discussion of the transversal index problem for foliations. Another application is given in [RT], where an almost Lie foliation structure is deformed to a Lie foliation structure by the heat flow method.

References


