

It might be allowed to mention that there is quite a number of misprints, but most of them are quite harmless like misspelling of names. It is more regrettable that the first definition of Hausdorff distance is blurred by a missing comma. That observation does not influence my opinion that this is a very good book for anyone interested in constructive function theory and that it certainly can be used as an educative graduate text-book.

#### REFERENCES

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*Introduction to the spectral theory of polynomial operator pencils*, by A. S. Markus. Translated by H. H. McFaden, *Translations of Mathematical Monographs*, vol. 71, American Mathematical Society, Providence, R.I., 1988, iv + 250 pp., \$95.00. ISBN 0-8218-4523-3

This book concerns the spectral theory of operator polynomials, i.e., of expressions of the form

$$(1) \quad A(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^n A_n,$$

where  $\lambda \in \mathbb{C}$  is a spectral parameter and  $A_0, \dots, A_n$  are operators acting in a Hilbert space  $H$ . This subject also includes the spectral theory of a single operator, which appears if one takes  $n = 1$ . The motivation for a polynomial spectral theory arises in the study of differential equations

$$(2) \quad \begin{cases} A_n \varphi^{(n)}(t) + \cdots + A_1 \varphi^{(1)}(t) + A_0 \varphi(t) = 0, \\ \varphi^{(j)}(0) = x_j, \quad j = 0, \dots, n-1. \end{cases}$$

Here the unknown function  $\varphi$  is a  $H$ -valued function on  $0 \leq t < \infty$  and the initial data  $x_0, \dots, x_{n-1}$  are vectors in  $H$ . A solution  $\varphi$  of (2) is called *elementary* whenever  $\varphi$  is of the form

$$(3) \quad \varphi(t) = e^{\lambda_0 t} \left( \sum_{\nu=0}^{k-1} \frac{1}{\nu!} t^\nu x_{k-1-\nu} \right).$$

If  $H = \mathbb{C}^m$  and the leading coefficient  $A_n$  is invertible, then any solution of (2) is a linear combination of elementary solutions. As is well known, the latter statement follows from the general fact that for  $A_n = I$  (the identity operator on  $H$ ) the function (3) is a solution of (2) if and only if

$$(A - \lambda_0) \hat{x}_0 = 0, \quad (A - \lambda_0) \hat{x}_j = \hat{x}_{j-1}, \quad j = 1, \dots, k-1,$$

where  $A$  is the so-called companion operator,

$$(4) \quad A = \begin{pmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & I \\ -A_0 & -A_1 & -A_2 & \dots & -A_{n-1} \end{pmatrix},$$

which has to be considered as an operator on  $H^n$ , the direct sum of  $n$  copies of  $H$ , and

$$\hat{x}_j = \begin{bmatrix} a_{0j} \\ a_{1j} \\ \vdots \\ a_{n-1,j} \end{bmatrix}, \quad a_{ij} = \sum_{\nu=0}^{\min(i,j)} \frac{i!}{(i-\nu)!} x_{j-\nu}, \quad j = 0, \dots, k-1.$$

In the infinite dimensional case, as is already clear from single operator theory, a solution of (2) does not have to be a linear combination of elementary solutions. In fact, it may happen that there are no elementary solutions at all, even if the coefficients are selfadjoint operators. An example of this kind is provided by the equation

$$\varphi^{(2)}(t) - (V + V^*)\varphi^{(1)}(t) + VV^*\varphi(t) = 0,$$

where  $V$  is the operator of integration, i.e.,

$$(Vf)(t) = 2i \int_0^t f(s) ds.$$

So the problem arises under which conditions do elementary solutions exist and, more importantly, when is the set of elementary solutions rich enough so that any solution of (2) can be expanded (in a neighborhood of zero) into a convergent series of elementary solutions. In a more modest form this leads to the problem of finding conditions which guarantee that the vector of initial conditions

$$\begin{pmatrix} \varphi(0) \\ \varphi'(0) \\ \vdots \\ \varphi^{(n-1)}(0) \end{pmatrix}$$

of each solution  $\varphi$  of (2) can be approximated by a linear combination of vectors of initial conditions of elementary solutions. The latter problem is the *problem of multiple completeness*. For  $A_n = I$  the multiple completeness problem is equivalent to the problem of completeness of the set of eigenvectors and generalized eigenvectors of the operator defined in (4).

The problem of multiple completeness was first introduced and studied by V. M. Keldysh in a four page note [1] which appeared in 1951 in Doklady. Keldysh identified natural classes of operator polynomials for which completeness holds, and he established some expansion theorems. Interesting results about the asymptotic behavior of eigenvalues also appeared in this note. Much later, in 1971, a detailed paper [2] of Keldysh, containing complete proofs, appeared in Uspehi. A translation into English

of Keldysh's 1951 Doklady note is an appendix in the present book. Full proofs of an important part of the Keldysh results were recovered before his 1971 paper appeared and found a place in the Gohberg-Kreĭn book [3]. Evaluations of the growth of the norm of the resolvent  $A(\lambda)^{-1}$  for  $\lambda \rightarrow \infty$  are the main tools in the proofs of the Keldysh results.

A second method for dealing with spectral problems for operator polynomials was developed in the middle of the sixties by M. G. Kreĭn and H. Langer [4, 5]. The Kreĭn-Langer method uses factorization as a tool. Consider the quadratic operator polynomial  $L(\lambda) = \lambda^2 I + \lambda A_1 + A_2$ , and let  $Z$  be a (right) operator root of  $L(\lambda) = 0$ , that is,

$$(5) \quad Z^2 + A_1 Z + A_0 = 0.$$

Then  $e^{tZ}x$  is a solution of the corresponding differential equation for each  $x \in H$ . One of the problems is to find operator roots  $Z_1$  and  $Z_2$  of  $L(\lambda) = 0$  so that

$$\phi(t) = e^{tZ_1}x_1 + e^{tZ_2}x_2,$$

where  $x_1$  and  $x_2$  run over all vectors in  $H$ , form the set of all solutions of  $L(d/dt)\phi = 0$ . Moreover, if such  $Z_1$  and  $Z_2$  have been found, one would like to know when the linear span of elementary solutions of the equations

$$\phi' = Z_1\phi, \psi' = Z_2\psi,$$

is dense (in a sense to be made more precise) in the space of all solutions. If  $Z$  is an operator root, then  $\lambda I - Z$  is a right divisor of  $A(\lambda)$ , that is

$$(6) \quad \lambda^2 I + \lambda A_1 + A_0 = (\lambda I - Z)(\lambda I - Y).$$

The method of Kreĭn-Langer uses the theory of spaces with an indefinite metric to obtain factorizations of the type (6) and to analyze their properties for the important case when  $A_0$  and  $A_1$  are selfadjoint. Here indefinite metrics turn up naturally, because the companion operator

$$\begin{pmatrix} 0 & I \\ -A_0 & -A_1 \end{pmatrix}$$

is selfadjoint in the indefinite metric

$$\left[ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right] := \langle A_1 x_1 + x_1, x_2 \rangle + \langle x_1, y_2 \rangle,$$

whenever  $A_0$  and  $A_1$  are selfadjoint relative to the usual inner product  $\langle \cdot, \cdot \rangle$  on  $H$ . The Kreĭn-Langer method is, therefore, of a geometric nature, while the one of Keldysh has an analytic character.

The set of problems considered in this book has important applications to partial differential equations and mathematical physics. For example, the spectral analysis of operator polynomials  $\lambda I - A - \lambda^2 B$ , with  $A$  and  $B$  compact and selfadjoint, appears in papers of S. G. Kreĭn and his students, which deal with problems of oscillation of a viscous fluid.

The type of problems and the approach proposed by Keldysh initiated a direction of research in which mainly Soviet mathematicians have taken part and which does not seem to be popular in the Western literature. It has produced a vast and rich literature. The Kreĭn-Langer approach has

led to two directions of research. One continued to use and developed further the geometric methods of operator theory in spaces with an indefinite metric with H. Langer as one of the main contributors. The other direction adapted Wiener-Hopf factorization methods for operator-valued functions and used these analytical tools to study the spectral properties of operator polynomials and analytic operator functions. Here the main contributors are the author of the present book and V. I. Macaev. Parallel, but with a stronger emphasis on the finite dimensional case, the second direction was also developed in the Western literature. In the latter, connections with mathematical systems theory and the theory of characteristic operator functions played an important role. (See, e.g., the papers Gohberg-Lancaster-Rodman, Kaashoek-Van der Mee-Rodman in the book's bibliography and the recent book [6] by Leiba Rodman.)

The book under review consists of four chapters. The first two (about 100 pp.) concern the main developments in the Keldysh theory and its applications. Chapter I deals with single nonselfadjoint operators which have discrete spectrum and are small perturbations of selfadjoint or normal operators. The main part of this chapter consists of the Keldysh theorems about completeness of eigenvectors and generalized eigenvectors of a single operator and the asymptotic behavior of the eigenvalues. In Chapter II the Keldysh theorems are extended to operator polynomials. For the completeness theorem two proofs are presented; one is based on linearization and the other follows the original Keldysh concept. Also, different versions of the Keldysh theorems are considered, including cases when the coefficients of the operator polynomials are unbounded.

The second part of the book (Chapters III and IV; again about 100 pp.) concerns the Wiener-Hopf factorization direction which originated from the Krein-Langer approach. This second part also contains an analysis of selfadjoint operator polynomials, but very little about connections with spaces with an indefinite metric. The first three sections of Chapter III are devoted to factorization theorems of Wiener-Hopf type for operator functions. They are used later to prove the existence of divisors with certain desired properties. The last chapter is dedicated to different classes of operator polynomials with selfadjoint coefficients. It contains the variational method. Connections with the original Krein-Langer method appear in the last section. All chapters have sections with applications to partial differential equations or to operator theory.

The author, who is one of the founders of this area in operator theory and who made fundamental contributions to this field, has produced a most valuable book. In a concise form (at a few places a little too concise perhaps) he has given an efficient account of the main developments during the last forty years. An introduction to the material of the first two chapters can also be found in the Gohberg-Krein book [3], but there is no other book that covers the material of the other two chapters. The book is a must for people working in operator theory, and it will be of interest to researchers in partial and ordinary differential equations and complex analysis. The book is not a text book, but parts of it could be used for seminar lectures or special topic courses.

