SMOOTH EXTENSIONS FOR A FINITE CW COMPLEX

GUIHUA GONG

The $C^*$-algebra extensions of a topological space can be made into an abelian group which is naturally equivalent to the $K$-homology group of odd dimension [1] which has a close relation with index theory and is one of the starting points of $KK$ theory [8].

The $C_p$-smoothness of an extension of a manifold was introduced in [3, 4], where $C_p$ denotes the Schatten-von Neumann $p$-class [5]. We generalize the notion of $C_p$-smoothness to a finite CW complex and obtain necessary and sufficient conditions for an extension of a finite CW complex to be $C_p$-smooth modulo torsion.

The notion of $C_p$-smooth extensions is one of the motivations for Connes' cyclic cohomology. In [2] Connes constructs a Chern map from $KK(C(M), C)$ to the cyclic cohomology of $C^\infty(M)$, and proves that this Chern map is a surjection modulo torsion. One consequence of the even counterpart of our main results is that this Chern map is a graded surjection modulo torsion. We will make this statement precise in Theorem 3.

Let $H$ be an infinite dimensional complex separable Hilbert space. By $L(H)$ and $K(H)$ we shall denote the $C^*$-algebra of bounded operators and compact operators on $H$, respectively, and $Q(H)$ will denote the quotient $L(H)/K(H)$ with canonical surjection $\pi : L(H) \rightarrow Q(H)$. For $X$ a compact metrizable space an extension $\tau \in \text{Ext}(X)$ of the algebra $C(X)$ by $K(H)$ is defined by a unital $*$ monomorphism $\tau : C(X) \rightarrow Q(H)$ [1].

**Definition 1.** Let $M$ be a smooth compact manifold (perhaps with boundary) and let $C^\infty(M)$ denote the $*$-algebra of all smooth functions on $M$. A $\tau \in \text{Ext}(M)$ is $C_p$-smooth if there exists a $*$-linear map $\rho : C^\infty(M) \rightarrow L(H)$ such that $\rho(ab) - \rho(a)\rho(b) \in C_p$ and $\pi \circ \rho = \tau|C^\infty(M)$.
This definition can be found in [2] and is equivalent to the definition in [4] by means of the $C^\infty$ functional calculus of Helton-Howe [6, 7].

In order to define $C_p$-smooth for a general finite CW complex, we shall use the following Lemma:

**Lemma.** If $X$ is a finite CW complex, then there exist a compact smooth manifold $M$ (perhaps with boundary), and two maps $f : X \to M$ and $g : M \to X$ such that $(g \circ f)$ is homotopic to $\text{id}|X$.

**Definition 2.** Let $X$, $M$ and $f$ be as in the Lemma. Then $\tau \in \text{Ext}(X)$ is $C_p$-smooth if $f_*\tau \in \text{Ext}(M)$ is $C_p$-smooth.

It is not difficult to prove that the $C_p$-smoothness does not depend on the choice of $M$ and the maps by using the following fact: Any continuous map between two smooth manifolds is homotopic to a smooth map. Similarly, we prove that the notion of $C_p$-smoothness of a manifold does not depend on the particular differential structure which answers the question on p. 68 of [3]. And also we prove that if $f : X \to Y$ is a continuous map between finite CW complexes $X$ and $Y$, then $f_*$ maps the $C_p$-smooth elements of $\text{Ext}(X)$ to the $C_p$-smooth elements of $\text{Ext}(Y)$.

Our main results are Theorems 1, 2, 3.

**Theorem 1.** Let $X$ be a finite CW complex, $X^k$ denote the $k$-skeleton of $X$, and $\tau \in \text{Ext}(X)$. Then there exists an integer $m_1 \neq 0$ such that $m_1 \tau$ is $C_n$-smooth if and only if there exists an integer $m_2 \neq 0$ such that $m_2 \tau \in i_*(\text{Ext}(X^{2n-1}))$, where $i_* : \text{Ext}(X^{2n-1}) \to \text{Ext}(X)$ is induced by the inclusion map $i : X^{2n-1} \to X$. Furthermore, if $X$ is a smooth compact $(2n-1)$-manifold, then each element in $\text{Ext}(X)$ is $C_p$-smooth when $p > n - \frac{1}{2}$.

The "only if" part of Theorem 1 generalizes the results in [3, 4]. It was shown in [3, 6] that the $C_1$-smooth elements of $\text{Ext}(X)$ come from the 1-skeleton modulo torsion. And also it was shown in [4] that each $C_{n-1}$-smooth element of $\text{Ext}(S^{2n-1})$ is trivial.

As a corollary of Theorem 1, we know that all the elements of $\text{Ext}(S^{2n-1})$ are $C_p$-smooth when $p > n - \frac{1}{2}$. This result solves the problem on p. 109 of [4]. As a special case, we have the following fact: If $(T_{z_1}, T_{z_2}, \ldots, T_{z_n})$ is the $n$-tuple of Toeplitz operators on $H^2(\partial B_n)$, then there exist $n$ compact operators $K_1, K_2, \ldots, K_n$ such that $[T_{z_i} + K_i, T_{z_j} + K_j] \in C_p$ and $[T_{z_i} + K_i, T_{z_j}^* + K_j^*] \in C_p$. 
when \( p > n - \frac{1}{2} \). There doesn’t seem to be any direct proof of this. The author does not know whether the elements of \( \text{Ext}(S^{2n-1}) \) are \( C_p \)-smooth when \( n - 1 < p \leq n - \frac{1}{2} \).

The following result is almost equivalent to Theorem 1 but is perhaps more useful in practice.

**Theorem 2.** Let \( X \) be a finite CW complex, \( \tau \in \text{Ext}(X) = K_1(X) \) and \( \text{ch} : K_1(X) \otimes \mathbb{C} \to H_{\text{odd}}(X, \mathbb{C}) \) be the Chern map, where \( H_{\text{odd}}(X, \mathbb{C}) \) denotes the direct sum of all the ordinary homology groups with complex coefficients of odd dimension. Then there exists an integer \( m \neq 0 \) such that \( m\tau \) is \( C_n \)-smooth if and only if

\[
\chi \tau \in \sum_{k=1}^n H_{2k-1}(X, \mathbb{C}).
\]

We also obtain some similar results about the \( p \)-summable Fredholm modules of \( C^\infty(M) \), which can be thought of as elements of \( K_0(M) = KK(C(M), \mathbb{C}) \), and about their Chern characters in the cyclic cohomology \( H_*(C^\infty(M)) \). In particular, we prove the following theorem.

**Theorem 3.** If \( M \) is a compact smooth manifold without boundary and \( \varphi \in H_*^k(C_\infty(M)) \) (\( \kappa \) even), then there exist \((k + 1)\) summable Fredholm modules \( \tau_i \) \((i = 1, 2, \ldots, n)\) and complex numbers \( \alpha_i \) \((i = 1, 2, \ldots, n)\) such that \( \sum_{i=1}^n \alpha_i \text{ch}^* \tau_i \sim \varphi \) in \( H_*^k(C_\infty(M)) \), where \( \text{ch}^* \) is Connes’ Chern map.

We would like to point out that A. Connes constructed the graded Chern characters

\[
\text{ch}^* : \{n + 1 \text{ summable Fredholm module}\} \to H_*^n(C_\infty(M))
\]

in §2 of [2], and that he also proved that

\[
\text{ch}^* : \{\text{finite summable Fredholm module}\} \to H_*^n(C_\infty(M))
\]

is surjective modulo torsion. Theorem 3 says that the Chern map is a graded surjection.

In order to prove our main theorems, we need some results from topology. Theorem 5 is a special case of the theorem on p. 210 line 7 of [9]. And Theorem 4 is perhaps also familiar to topologists. We provide an outline of a proof for Theorem 4 since we have been unable to find a precise reference.

**Theorem 4.** Let \( X \) be compact metrizable space. For any \( \tau \in K^1(X) \), there exist maps \( f_i : X \to S^{2n-1} \) \((i = 1, 2, \ldots, k)\) such that \( m\tau = \sum_{i=1}^k f_i^* \theta_i \) for some integer \( m \), where \( \theta_i \) is the canonical generator of \( K^1(S^{2n-1}) \).

**Theorem 5.** If \( X \) is a finite CW complex and \( \tau \in H_k(X) \), then there exist a smooth compact oriented \( k \)-manifold \( M \) without
boundary and a map \( f : M \to X \) such that \( m\tau = f_*\theta \) for some integer \( m \neq 0 \) and \( \theta \in H_k(M) \).

To prove Theorem 4, we only need to prove the case of \( X = U(n) \) because each element of \( K^1(X) \) can be realized as the pullback of an element in \( K^1(U(n)) \) via a map from \( X \) to \( U(n) \). The idea is to use obstruction theory and a result about Whitehead products [10, Theorem 8.9] to construct two maps: \( u : S^1 \times S^3 \times \cdots \times S^{2n-1} \to U(n), \ v : U(n) \to S^1 \times S^3 \times \cdots \times S^{2n-1} \), such that

\[
(v \circ u)^* : K^1(S^1 \times S^3 \times \cdots \times S^{2n-1})
\]

\[
= Z^{2n-1} \to K^1(S^1 \times S^3 \times \cdots \times S^{2n-1})
\]

can be represented by a matrix

\[
\begin{pmatrix}
 m_1 & 0 & \cdots & 0 \\
 0 & m_2 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & m_{2n-1}
\end{pmatrix}
\]

where \( m_k \neq 0 \) are integers, and \((u \circ v)^* : K^1(U(n)) = Z^{2n-1} \to K^1(U(n))\) has the same form as \((v \circ u)^*\). Then we can reduce the problem to the case of \( S^1 \times S^3 \times \cdots \times S^{2n-1} \) which can be easily done.

Using Proposition 3 in [4] and Theorem 4, we can prove the “only if” part of Theorem 1. For the “if” part we use Theorem 5.

The even counterpart of Theorems 1, 2 can be obtained in a similar manner and this is used in proving Theorem 3.

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Department of Mathematics, State University of New York at Stony Brook, Stony Brook, New York 11794