A GENERALIZATION OF SELBERG'S BETA INTEGRAL

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ABSTRACT. We evaluate several infinite families of multidimensional integrals which are generalizations or analogs of Euler's classical beta integral. We first evaluate a $q$-analog of Selberg's beta integral. This integral is then used to prove the Macdonald-Morris conjectures for the affine root systems of types $S(C_j)$ and $S(C_j)^\vee$ and to give a new proof of these conjectures for $S(BC_j)$, $S(B_j)$, $S(B_j)^\vee$ and $S(D_j)$.

1. INTRODUCTION

In 1944, A. Selberg [23] evaluated the following integral (see also Aomoto [1]):

\begin{equation}
\int_0^1 \cdots \int_0^1 \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2z} \prod_{j=1}^n t_j^{x_j - 1} (1 - t_j)^{y_j - 1} dt_j
= \prod_{j=1}^n \frac{\Gamma(x + (j - 1)z)\Gamma(y + (j - 1)z)\Gamma(jz + 1)}{\Gamma(x + y + (n + j - 2)z)\Gamma(z + 1)},
\end{equation}

where $n$ is a positive integer, $x, y, z \in \mathbb{C}$ and $\text{Re}(x), \text{Re}(y) > 0$ and $\text{Re}(z) > -\max\{\frac{1}{n}, \text{Re}(x)/(n-1), \text{Re}(y)/(n-1)\}$. For $n = 1$, the integral (1) reduces to Euler's classical beta integral.

Now let $n \geq 1$ and $a_1, a_2, a_3, a_4, b, q \in \mathbb{C}$ with

$$\max\{|a_1|, \ldots, |a_4|, |b|, |q| < 1.$$  

For $c \in \mathbb{C}$ define

$$[c; q]_\infty = [c]_\infty = \prod_{k=0}^{\infty} (1 - cq^k).$$

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If $T^n$ is the $n$-fold direct product of the unit circle \( \{ t \in \mathbb{C} | |t| = 1 \} \) traversed in the positive direction, then we can evaluate the integral

\[
\frac{1}{(2\pi i)^n} \int_{|z| = 1} \prod_{1 \leq j < k \leq n} \frac{[t_j t_k^{-1}]_\infty [t_j^{-1} t_k]_\infty [t_j t_k]_\infty [t_j^{-1} t_k^{-1}]_\infty}{[b t_j t_k^{-1}]_\infty [b t_j^{-1} t_k^{-1}]_\infty [b t_j t_k]_\infty [b t_j^{-1} t_k]_\infty} dt_j
\]

\[
\cdot \prod_{j=1}^{n} \frac{[t_j^2]_\infty [t_j^{-2}]_\infty}{\prod_{k=1}^{n} \{a_k t_j \}_\infty [a_k t_j^{-1}]_\infty} t_j
\]

\[
= 2^n n! \prod_{j=1}^{n} \frac{[b]_\infty [b^{n+j-2}]_\infty}{\prod_{1 \leq k \leq 4} \{a_k q b^{j-1}\}_\infty}.
\]

Then $n = 1$ case of integral (2) is due to Askey and Wilson [4]. The integral (2) is a $q$-analog of (1) in the sense that after a change of variables and an appropriate specialization of (2) and limit as $q \to 1$, then (1) can be deduced from (2).

Selberg's integral (1) has had diverse applications in fields ranging from number theory, physics, statistics, combinatorics, algebra and analysis. Two particular applications were a use by Bombieri to prove Mehta's conjecture [18] and by Macdonald [17] to prove some of his conjectures ($q = 1$ case) for the affine root systems (for definition and properties see [15]) of types $S(BC_1), S(B_2), S(B_3)^\vee, S(C_1), S(C_2)^\vee$ and $S(D_l)$ for all $l \geq 1$ (when defined). Just as Macdonald used integral (1) to prove some of his ($q = 1$) conjectures, we will use integral (2) to prove for the same set of affine root systems the corresponding Macdonald-Morris conjectures with arbitrary parameter $q$.

Macdonald's root system conjectures in [17] were motivated partly by a conjecture of Dyson [7] related to the root system $A_n$, a $q$-analogue of Dyson's conjecture made by Andrews [2] and some conjectures of Morris [19] for the root system of type $G_2$. Dyson's conjecture was proved by Gunson [10] and Wilson [25]. The Andrews-Dyson conjecture was proved by Zeilberger and Bressoud [28].

Morris' Conjecture A in [19] for arbitrary parameter $q$ and any reduced irreducible affine root system $S$ extends Macdonald's Conjectures 2.3 and 3.1 in [17]. In the simplest case of these Macdonald-Morris conjectures, let $R$ be a reduced finite (not affine) root system of rank $l$ with basis \( \{ \alpha_1, \ldots, \alpha_l \} \). For each $\alpha \in R$, let $e^\alpha$ be the formal exponential, which is an element of the group ring of the lattice generated by $R$. Let $d_1, \ldots, d_l$
be the degrees of the fundamental invariants of the Weyl group $W(R)$.

**Conjecture (Macdonald [17, Conjecture 3.1]).** With the above notation, the constant term (i.e. involving $q$ but no exponential $e^\alpha$) in

$$\prod_{\alpha > 0} \prod_{i=1}^{k} (1 - q^{i-1} e^{-\alpha})(1 - q^{i} e^{\alpha})$$

where $k$ is a positive integer or $\infty$ is

$$\prod_{i=1}^{l} \left\lfloor \frac{kd_i}{k} \right\rfloor$$

where

$$\binom{n}{r}$$

is the "q-binomial coefficient"

$$\frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-r+1})}{(1 - q)(1 - q^2) \cdots (1 - q^r)}.$$

We will actually prove the more general Morris' Conjecture A [19] for the affine root systems $S$ of types $S(BC_l), S(B_l), S(B_l)^\vee, S(C_l), S(C_l)^\vee$ and $S(D_l)$ for all $l \geq 1$ (when defined) and for arbitrary parameter $q$. Macdonald's Conjecture 3.1 stated above, where $R$ is a finite root system of type $B_l, C_l$ or $D_l$, then follows as a special case of Morris' Conjecture A for $S(B_l), S(C_l)$ and $S(D_l)$. Kadell [14] has previously proved these conjectures for all affine root systems of type $S(BC_l)$, and hence $S(B_l), S(B_l)^\vee$ and $S(D_l)$. The Macdonald-Morris conjectures for $R = G_2$ have been proved by Habsieger [13] and Zeilberger [26]. See Garvan [8] for $F_4$, Garvan and Gonnet [9] for $S(F_4)^\vee$, Zeilberger [27] for $S(G_2)^\vee$ and Opdam [20] for the $q = 1$ conjectures. There is also the conjecture of Rahman [21] which seems related to the special case of integral (2) where $a_2 = q^{1/2}a_1$ and $a_4 = q^{1/2}a_3$.

### 2. Proof of integral (2)

Since the $n = 1$ case of (2) is proved in [4], we may assume that $n \geq 2$. Denote the integral on the left-hand side of (2) by $I_n(a_1, a_2, a_3, a_4; b; q)$. Let $c_j \in C$, $|c_j| < 1$, for $1 \leq j \leq 2n + 2$
with \( q \) and \( T \) as above. In [11] we have evaluated the integral

\[
\frac{1}{(2\pi i)^n} \int_{T^n} \frac{\prod_{1 \leq j < k \leq n} \{ [t_j t_k^{-1}] \}_\infty [t_j^{-1} t_k] \_\infty [t_j t_k] \_\infty [t_j^{-1} t_k^{-1}] \_\infty }{\prod_{j=1}^{2n+2} \prod_{k=1}^{n} [c_j t_k] \_\infty [c_j t_k^{-1}] \_\infty } \cdot \prod_{j=1}^{n} \frac{[t_j^2] \_\infty [t_j^{-2}] \_\infty}{t_j} \, dt_j
\]

\[
= 2^n n! \left[ \prod_{j=1}^{2n+2} c_j \right] \_\infty \prod_{1 \leq j < k \leq 2n+2} [c_j c_k] \_\infty .
\]

With notation as above, consider the integral

\[
\frac{1}{(2\pi i)^{2n-1}} \int_{T^n} \int_{T^{n-1}} \frac{\prod_{1 \leq j < k \leq n} \{ [t_j t_k^{-1}] \}_\infty [t_j^{-1} t_k] \_\infty [t_j t_k] \_\infty [t_j^{-1} t_k^{-1}] \_\infty }{\prod_{j=1}^{n} \prod_{k=1}^{4} [a_k t_j] \_\infty [a_k t_j^{-1}] \_\infty } \cdot \prod_{j=1}^{n} \prod_{k=1}^{n} \{ [s_j s_k^{-1}] \}_\infty [s_j^{-1} s_k] \_\infty [s_j s_k] \_\infty [s_j^{-1} s_k^{-1}] \_\infty } \cdot \prod_{j=1}^{n} \prod_{j=1}^{n} \{ [b^{1/2} s_k t_j] \}_\infty [b^{1/2} s_k^{-1} t_j] \_\infty [b^{1/2} s_k t_j] \_\infty [b^{1/2} s_k^{-1} t_j] \_\infty } \cdot \prod_{j=1}^{n} \frac{[s_j^2] \_\infty [s_j^{-2}] \_\infty}{s_j} \, ds_j \prod_{j=1}^{n} \frac{dt_j}{t_j}
\]

where \( b^{1/2} \) is any fixed square root of \( b \). In the integral (4) we may use identity (3) to evaluate the interior integral either with respect to the set of variables \( \{ s_1, \ldots, s_{n-1} \} \) or, by changing the order of integration, with respect to the set of variables \( \{ t_1, \ldots, t_n \} \). Equating the resulting integrals we obtain

\[
\frac{2^{n-1} (n-1) ![b^n] \_\infty}{[q]^{n-1} [b] \_\infty} I_n(a_1, a_2, a_3, a_4; b; q)
\]

\[
= 2^n n! [b^{n-1} \prod_{j=1}^{4} a_j] \_\infty \prod_{1 \leq j < k \leq 4} [a_j a_k] \_\infty I_{n-1}(a_1 b^{1/2}, \ldots, a_4 b^{1/2}; b; q).
\]
We finish the proof of identity (2) by doing induction on \( n \), using identity (5) and the Askey-Wilson integral for the case \( n = 1 \).

3. Morris' Conjecture A

We sketch a proof of Morris' Conjecture A [19] for the affine root systems \( S \) of types \( S(BC_l) \), \( S(B_l) \), \( S(C_l) \), \( S(C_l)_\vee \) and \( S(D_l) \) where \( l \geq 1 \) (when defined) and for arbitrary parameter \( q \). The proof consists of specializing the parameters in identity (2) and making use of the identity found in Theorem 2.8 of [16]. As an illustration of this method of proof of Morris' Conjecture A, consider the case \( S = S(C_l) \) where \( l \geq 2 \). Consider the integral \( I_l(a^{1/2}, -a^{1/2}, q^{1/2}a^{1/2}, -q^{1/2}a^{1/2}; b; q) \) where \( |a|, |b| < 1 \). Multiply the integrand in this integral by

\[
\prod_{1 \leq j < k \leq l} \frac{(1 - bw(t^{-1}_j t_k))(1 - bw(t^{-1}_j t_k))}{(1 - w(t^{-1}_j t_k))(1 - w(t^{-1}_j t_k))} \prod_{j=1}^l \frac{(1 - aw(t^{-2}_j))}{(1 - w(t^{-2}_j))},
\]

where \( w \) is an element of the Weyl group \( W \) of \( C_l \), i.e. a permutation of the variables \( t_1, \ldots, t_l \) together with inversions \( t_j \to t_j^{-1} \) and the corresponding action on \( t_1^{-1}, \ldots, t_l^{-1} \). The resulting integral is independent of \( w \in W \). Now summing over \( w \in W \) and using the identity [16, Theorem 2.8] for \( C_l \) we obtain

\[
(6) \quad \frac{1}{(2\pi i)^l} \int_{T^l} \prod_{1 \leq j < k \leq l} \frac{[t_j t_k^{-1}]_\infty [qt_j^{-1} t_k]_\infty [qt_j^{-1} t_k]_\infty [qt_j^{-1} t_k]_\infty}{[bt_j t_k^{-1}]_\infty [qt_j^{-1} t_k]_\infty [qt_j^{-1} t_k]_\infty [qt_j^{-1} t_k]_\infty} \prod_{j=1}^l \frac{[r_j^2]_\infty [qt_j^{-2}]_\infty dt_j}{[at_j^2]_\infty [qt_j^{-2}]_\infty t_j}
\]

\[
= \prod_{j=1}^l \frac{[q b]_\infty [q a^2 b^{1+j-2}]_\infty [qa b^{j-1}]^2_\infty}{[q]_\infty [q b^j]_\infty [qa^2 b^{2(j-1)}]^2_\infty},
\]

which is equivalent to Morris' Conjecture A for \( S(C_l) \) [19, p. 131]. Setting \( a = b \) in (6), this also proves Macdonald's Conjecture 3.1 for \( R = C_l \) as stated above.

4. Some integral evaluations

We state some integral identities whose proofs are similar to that of (2), making use of integral identities from [11 and 12].
Details of the proofs of these and related integral identities should be given elsewhere.

Let \( n \geq 1 \) and \( z_1, \ldots, z_n, \alpha_1, \ldots, \alpha_4, a_1, \ldots, a_4, \beta_1, \beta_2, b, \delta \in \mathbb{C} \) and \( m_1, \ldots, m_n \in \mathbb{Z} \). Choose \( z_1, \ldots, z_n \) so that the integrands in the integrals (9) and (10) below have no poles. Then

\[
(7) \quad \frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \prod_{1 \leq j < k \leq n} \left\{ \frac{\Gamma(\delta + t_j - t_k)}{\Gamma(t_j - t_k)\Gamma(t_k - t_j)} \right\} \prod_{j=1}^{n} \frac{\Gamma(\delta + t_j)\Gamma(\delta - t_j)}{\Gamma(2t_j)\Gamma(-2t_j)}
\]

\[
= 2^n n! \prod_{j=1}^{n} \frac{\Gamma(j\delta)}{\Gamma(\delta)\Gamma((n + j - 2)\delta + \sum_{k=1}^{4} a_k)},
\]

where the contours of integration are the imaginary axis and

\[
\min\{\text{Re}(\delta), \text{Re}(\alpha_1), \ldots, \text{Re}(\alpha_4)\} > 0;
\]

\[
(8) \quad \frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \prod_{1 \leq j < k \leq n} \frac{\Gamma(\delta + t_j - t_k)}{\Gamma(t_j - t_k)} \prod_{j=1}^{n} \left\{ \prod_{k=1}^{2} [(\Gamma(\alpha_k + t_j)\Gamma(\beta_k - t_j))] dt_j \right\}
\]

\[
= n! \prod_{j=1}^{n} \frac{\Gamma(j\delta)}{\Gamma(\delta)\Gamma((n + j - 2)\delta + \sum_{k=1}^{4} (\alpha_k + \beta_k))},
\]

where the contours of integration are the imaginary axis and

\[
\min\{\text{Re}(\delta), \text{Re}(\alpha_1), \text{Re}(\alpha_2), \text{Re}(\beta_1), \text{Re}(\beta_2)\} > 0;
\]
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\[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_{1 \leq j, k \leq n, j \neq k} \frac{\Gamma(1 + z_j + t_j - z_k - t_k)}{\Gamma(1 + \delta + z_j + t_j - z_k - t_k)} \]

\[\cdot \prod_{j=1}^{n} \prod_{k=1}^{\frac{2}{\Gamma(1 + \alpha_k + z_j + t_j)} \Gamma(1 + \beta_k - z_j - t_j)} \]

\[\cdot \prod_{j=1}^{\infty} \frac{\Gamma(1 + \delta) \Gamma(1 + (n + j - 2)\delta + \sum_{k=1}^{2} (\alpha_k + \beta_k))}{\Gamma(1 + j\delta) \prod_{k, l=1}^{2} \Gamma(1 + \alpha_k + \beta_l + (j - 1)\delta)} \]

\[= \begin{cases} 
\prod_{j=1}^{\infty} & 	ext{if } m_1 = \cdots = m_n = 0 \\
0 & \text{otherwise}
\end{cases} \]

where

\[\min \left\{ \text{Re} \left( (n + 1)\delta + \sum_{k=1}^{2} (\alpha_k + \beta_k) \right), \right. \]

\[\left. \text{Re} \left( 2(n - 1)\delta + \sum_{k=1}^{2} (\alpha_k + \beta_k) \right) \right\} > -1; \]

\[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_{1 \leq j, k \leq n} \left\{ \frac{[bq^{1+z_j+t_j-z_k}]_{\infty}[bq^{1-z_j-t_j+z_k+t_k}]_{\infty}}{[q^{1+z_j+t_j-z_k}]_{\infty}[q^{1-z_j-t_j+z_k+t_k}]_{\infty}} \cdot \prod_{k=1}^{4} \frac{[a_kq^{1+z_j+t_j}]_{\infty}[a_kq^{1-z_j-t_j}]_{\infty}}{[q^{1+2z_j+2t_j}]_{\infty}[q^{1-2z_j-2t_j}]_{\infty}} \right\} \cdot e^{2\pi i m_1 t} dt_j \]

\[= \begin{cases} 
\prod_{j=1}^{\infty} & \text{if } m_1 = \cdots = m_n = 0 \\
0 & \text{otherwise}
\end{cases} \]
where
\[ \max \left\{ \left| q b^{n-1} \prod_{k=1}^{4} a_k \right|, \left| q b^{2(n-1)} \prod_{k=1}^{4} a_k \right| \right\} < 1 \]
and for simplicity we assume that \( q \in \mathbb{R}, \ 0 < q < 1 \). The \( n = 1 \) case of (7) is due to de Branges [6] and Wilson [24], of (8) to Barnes [5], of (9) to Ramanujan [22] and (10) essentially to Askey [3].

Remarks. The integrals (9) and (10) are equivalent to multiple series summation theorems which generalize classical bilateral hypergeometric series summation theorems: Dougall’s \( 2H_2 \) sum and Bailey’s \( \psi_6 \) sum. A similar connection between some related integral evaluations and the corresponding multiple series identities is explained in [12]. As we plan to describe elsewhere, we are led to conjecture a family of multiple series summation identities which are equivalent to the Macdonald-Morris conjectures and contain the Macdonald identities [15] as special cases.

References

3. R. Askey, Beta integrals and q-extensions, Annamalai Univ. lecture, preprint.
9. F. Garvan and G. H. Gonnet, A proof of the two parameter q-case of the Macdonald-Morris root system conjecture for \( S(F_4) \) and \( S(F_4) \) via Zeilberger’s method, preprint.

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