A SHARP COUNTEREXAMPLE ON THE REGULARITY OF $\Phi$-MINIMIZING HYPERSURFACES

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A standard problem in the calculus of variations seeks a hypersurface $S$ of least area bounded by a given $(n-2)$-dimensional compact submanifold of $\mathbb{R}^n$. More generally, given any smooth norm $\Phi$ on $\mathbb{R}^n$, seek to minimize

$$\Phi(S) = \int_S \Phi(n),$$

where $n$ is the unit normal vector to $S$. Think of the integrand $\Phi$ as assigning a cost or energy to each direction. We assume that $\Phi$ is elliptic (uniformly convex), the standard hypothesis for regularity.

Geometric measure theory (cf. [M, Chapters 5, 8], [F 1, 5.1.6, 5.4.15]) guarantees the existence of a (possibly singular) $\Phi$-minimizing hypersurface with given boundary. For the case of area ($\Phi(n) = 1$), area-minimizing hypersurfaces are regular embedded manifolds up through $\mathbb{R}^2$, but sometimes have singularities in $\mathbb{R}^8$ and above. For general elliptic $\Phi$, a result of Almgren, Schoen, and Simon [Alm S S, Theorem II.7] guarantees regularity up through $\mathbb{R}^3$, but there were no examples of singularities below $\mathbb{R}^8$. We establish the sharpness of the Almgren–Schoen–Simon regularity result by giving a singular $\Phi$-minimizing hypersurface in $\mathbb{R}^3$.

The surface is the cone $C$ over the Clifford torus $S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$:

$$C = \{ (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x| = |y| \leq 1 \}.$$

The norm $\Phi$ depends smoothly on $\theta = \tan^{-1}(|y|/|x|)$ alone, so that we may view $\Phi$ as a norm on $\mathbb{R}^2$. The unit $\Phi$-ball is pictured in Figure 1. Any smooth, symmetric, uniformly convex approximation of the square will do. Note that $\Phi$ is smaller (say 1) on...
Figure 1. The unit \( \Phi \)-ball. The smallness of \( \Phi \) in the diagonal directions helps to make the cone \( C \) \( \Phi \)-minimizing.

The normals to diagonal directions, which occur in the cone \( C \), so that \( \int_C \Phi(\mathbf{n}) \) is relatively small.

The unit ball of the norm dual to \( \Phi \), called the Wulff crystal \( W(\Phi) \), is pictured in Figure 2. The Wulff crystal itself solves an important problem: its boundary surface \( S \) minimizes \( \Phi(S) \) for fixed volume enclosed (cf. [T, §1]). In nature \( \Phi(S) \) represents the surface energy of a crystal, and the Wulff crystal \( W(\Phi) \) gives the shape which a fixed volume of material assumes to minimize surface energy. The Wulff crystal of our norm \( \Phi \) resembles a pivalic acid crystal (see Figure 3).

The Proof. The proof that the cone \( C \) over \( S^1 \times S^1 \subset \mathbb{R}^4 \) is \( \Phi \)-minimizing employs the "method of calibrations" (cf. [HL, Introduction]). One must produce a closed differential 3-form or "calibration" \( \varphi \) such that for any point \( p \) and unit 3-plane \( \xi \), with unit normal \( \xi \),

\[
\langle \xi, \varphi(p) \rangle \leq \Phi(\xi),
\]

with equality whenever \( \xi \) is the oriented unit tangent to \( C \) at \( p \).

Then if \( S \) is any other surface with the same boundary,

\[
\Phi(C) = \int_C \varphi = \int_S \varphi \leq \Phi(S),
\]

so that \( C \) is \( \Phi \)-minimizing.

Finding a calibration \( \varphi \) remains an art, not a science. Our calibration is

\[
\varphi = (\sin^2 2\theta \, dr + \sin 4\theta \, r \, d\theta) \wedge d\theta_1 \wedge d\theta_2,
\]
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**Figure 2.** The Wulff crystal $W(\Phi)$ may be defined as the unit ball for the norm dual to $\Phi$. For fixed volume, $W(\Phi)$ has the least surface energy measured by $\Phi$.

**Figure 3.** The pivalic acid crystal resembles the Wulff crystal $W(\Phi)$ of $\Phi$ [GS].

where $r^2 = |x|^2 + |y|^2$, $\theta = \tan^{-1}(|y|/|x|)$, $\theta_1 = \arg x$, $\theta_2 = \arg y$.

It resembles the 7-form of H. Federer's proof [F2, §6.3] after H. B. Lawson [L, §5] that the cone over $S^3 \times S^3$ is area-minimizing. At any point in our cone $C$, $\theta = \pi/4$, and $\varphi(\pi/4) = dr \wedge d\theta_1 \wedge d\theta_2$ is precisely dual to each unit tangent $\xi_0$ to $C$. Hence

$$\langle \xi, \varphi(p) \rangle \leq 1 \leq \Phi^*(\xi),$$
with equality whenever $\xi = \xi_0$. Thus (1) holds at points $p \in C$. Unfortunately, for $p \notin C$ (for example $\theta = \pi/8$), the $\sin 4\theta$ term, which is necessary to make $\varphi$ closed, tends to make $\varphi$ big. In order for (1) to hold, the largeness of $\varphi(\pi/8)$ must be somehow compensated for by the largeness of $\Phi(\pi/8)$.

Establishing the estimate (1) at all points almost always is a main difficulty.

For the case of area, the right-hand side is 1, and the estimate becomes $|\varphi(p)| \leq 1$, independent of $\xi$. For a general integrand $\Phi$, the estimate involves both $p$ and $\xi$. This difficulty explains why calibrations have not been applied specifically to integrands other than area before.

We handle this difficulty with a lemma that associates with $\varphi$ the function on unit vectors

$$G(w) = \sup\{|\varphi(p)|: w \text{ is the oriented unit normal to the } (n-1)-\text{plane dual to } \varphi(p)\}.$$ 

The lemma says that the desired estimate (1) holds if the graph of $G$ lies inside the Wulff crystal $W(\Phi)$, thus reducing the required estimate to a single parameter.

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**References**


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