Sierpiński, W., *Leçons sur les nombres transfinis*, Monographies Emile Borel, Paris, 1928.

Szpilrajn, E., *Sur l'extension de l'ordre partiel*, Fund. Math. **16** (1930), 386–389.

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§1. INTRODUCTION

During the past decades the *mathematical* theory of nonlinear three-dimensional elasticity has undergone a considerable renewed interest, reflected for instance by the books of Marsden and Hughes [17], Ciarlet [8], and the book reviewed here.

The existence results available at the present time fall in two categories:

In one approach (described in §§2 and 5) the problem is posed as a *system of three quasilinear partial differential equations of the second order*, together with specific boundary conditions (cf. (13)), and one tries to obtain "*local*" *existence results based on the implicit function theorem*; this approach, which was initiated by Stoppelli [18], is the central theme of the book under review.

In another approach (described in §§3 and 4), the problem is posed as a *minimization problem for the associated energy* (cf. (20)), and one tries to adapt the paraphernalia of the calculus of variations (infimizing sequences, weak convergence, weak lower semi-continuity, etc.) to this problem, which is "highly nonconvex"; this approach is the basis of a famous existence result of Ball [3].

All these results apply to "static" equilibria, i.e. to problems that are *time-independent*. While substantial progress has thus been made in the study of statics, the mathematical analysis of time-dependent three-dimensional elasticity still meets with inextricable difficulties. The proofs of the available existence results "for large

times," even though they apply to only one space variable, are already exceedingly delicate.

§2. Three-dimensional elasticity

Detailed expositions of the mathematical modeling of three-dimensional elasticity are found in Truesdell and Noll [19], Wang and Truesdell [23], Gurtin [14], Marsden and Hughes [17, Chapters 1–5], Ciarlet [8, Chapters 1–5].

The central problem in nonlinear, three-dimensional, static elasticity consists in finding the equilibrium position of an elastic body when it is subjected to applied forces. This body occupies a *reference configuration* $\overline{\Omega}$ in the absence of forces, where Ω is a domain in $\mathbf{R}^3$, i.e. a bounded, connected, open subset of $\mathbf{R}^3$ with a Lipschitz-continuous boundary Γ; in particular then, a unit normal vector $\mathbf{n} = (n_i)$ exists almost everywhere along Γ.

When subjected to *applied forces*, the body occupies a *deformed configuration* $\boldsymbol{\varphi}(\overline{\Omega})$, where the mapping $\boldsymbol{\varphi}: \overline{\Omega} \to \mathbf{R}^3$, which is called a *deformation*, must be orientation-preserving in the set $\overline{\Omega}$ and injective on the set Ω, in order to be physically acceptable (the reason a deformation need not be injective on $\overline{\Omega}$ is that self-contact must be allowed).

Let $\mathbf{M}^3$ denote the set of all real matrices of order 3 and let

$$\mathbf{M}^3_+ = \{\mathbf{F} \in \mathbf{M}^3;\ \det \mathbf{F} > 0\}.$$

Then the orientation-preserving character of a deformation imposes that its deformation gradient $\nabla\boldsymbol{\varphi}(x)$, defined by

$$\nabla\boldsymbol{\varphi} = \begin{pmatrix} \partial_1\varphi_1 & \partial_2\varphi_1 & \partial_3\varphi_1 \\ \partial_1\varphi_2 & \partial_2\varphi_2 & \partial_3\varphi_2 \\ \partial_1\varphi_3 & \partial_2\varphi_3 & \partial_3\varphi_3 \end{pmatrix}, \qquad \text{where } \partial_i = \frac{\partial}{\partial x_i},$$

be in the set $\mathbf{M}^3_+$ for all $x \in \overline{\Omega}$.

A body occupying a deformed configuration $\boldsymbol{\varphi}(\overline{\Omega})$, and subjected to applied body forces in its interior and to applied surface forces on a portion $\boldsymbol{\varphi}(\Gamma_1)$ of its boundary, where Γ_1 is a subset of Γ, is in *static equilibrium* if the fundamental *stress principle of Euler and Cauchy* is satisfied. This axiom implies the celebrated *Cauchy theorem*, according to which:

(i) There exists a tensor field $\mathbf{T}: \overline{\Omega} \to \mathbf{M}^3$ that satisfies the *equilibrium equations over the reference configuration*:

$$(1) \qquad \begin{cases} -\mathbf{div}\,\mathbf{T}(x) = \hat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)), & x \in \Omega, \\ \mathbf{T}(x)\mathbf{n}(x) = \hat{\mathbf{g}}(x, \nabla\boldsymbol{\varphi}(x)), & x \in \Gamma_1. \end{cases}$$

or, componentwise,

$$(2)\qquad \begin{cases} -\sum_{j=1}^{3} \partial_j T_{ij}(x) = \hat{f}_i(x, \boldsymbol{\varphi}(x)), & x \in \Omega,\ 1 \le i \le 3, \\ \sum_{j=1}^{3} T_{ij}(x) n_j(x) = \hat{g}_i(x, \nabla\boldsymbol{\varphi}(x)), & x \in \Gamma_1,\ 1 \le i \le 3. \end{cases}$$

(ii) The tensor $\mathbf{T}(x) = (T_{ij}(x))$, which is called the *first Piola–Kirchhoff stress tensor* at the point $x \in \Omega$, is the *Piola transform*

$$(3)\qquad \mathbf{T}(x) = (\det \nabla\boldsymbol{\varphi}(x))\mathbf{T}^{\boldsymbol{\varphi}}(\boldsymbol{\varphi}(x))\nabla\boldsymbol{\varphi}(x)^{-T}$$

of the *Cauchy stress tensor* $\mathbf{T}^{\boldsymbol{\varphi}}(\boldsymbol{\varphi}(x))$ at the point $\boldsymbol{\varphi}(x)$; since the Cauchy stress tensor is symmetric, one thus has, by (3),

$$(4)\qquad \mathbf{T}(x)\nabla\boldsymbol{\varphi}(x)^{T} \in \mathbf{S}^3 \qquad \text{for all } x \in \overline{\Omega},$$

where $\mathbf{S}^3 = \{\mathbf{A} \in \mathbf{M}^3;\ \mathbf{A} = \mathbf{A}^T\}$.

The mappings $\hat{\mathbf{f}}\colon \Omega \times \mathbf{R}^3 \to \mathbf{R}^3$ and $\hat{\mathbf{g}}\colon \Omega \times \mathbf{M}^3_+ \to \mathbf{R}^3$ respectively measure the *density of the applied body force* per unit volume, and the *density of the applied surface force* per unit area, in the reference configuration. For instance, a *pressure load* corresponds to a density $\hat{\mathbf{g}}$ of the form

$$(5)\qquad \hat{\mathbf{g}}(x, \nabla\boldsymbol{\varphi}(x) = -\pi(\det \nabla\boldsymbol{\varphi}(x))\nabla\boldsymbol{\varphi}(x)^{-T}\mathbf{n}(x), \qquad x \in \Gamma_1,$$

where π is a real constant, called the *pressure*; the *gravity field* corresponds to a density $\hat{\mathbf{f}}$ of the form

$$(6)\qquad \hat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)) = -g\rho(x)\mathbf{e}_3, \qquad x \in \Omega,$$

where g is the gravitational constant, $\rho(x)$ is the mass density, and $\mathbf{e}_3$ is the "third" basis vector, assumed to be "vertical" and "upward oriented." The density in (6) is that of a "dead load": an applied body or surface force is a *dead load* if its density is independent of the particular deformation $\boldsymbol{\varphi}$ considered. Note in passing that actual applied forces can seldom be modeled as dead loads (except the gravity field, or a pressure load with $\pi = 0!$).

The equations of equilibrium (1) must clearly be complemented by equations that specify the nature of the constituting material that is considered; the undetermination of equations (1) is also clear from a mathematical standpoint, since there are *three* equations in (2) and *nine* unknowns, the three components of the deformation and the six independent components of the tensor $\mathbf{T}$ (there are three equations in relation (4)).

A material is *elastic* if, at each point $\boldsymbol{\varphi}(x)$ of the deformed configuration, the Cauchy stress tensor $\mathbf{T}^{\boldsymbol{\varphi}}(\boldsymbol{\varphi}(x))$ is solely a function of x and of the deformation gradient $\nabla\boldsymbol{\varphi}(x)$. Equivalently, by

(3), a material is elastic if, at each point $x \in \overline{\Omega}$, the first Piola–Kirchhoff stress tensor is expressed in terms of x and $\nabla\boldsymbol{\varphi}(x)$ through a *constitutive equation* of the form

$$\begin{aligned} &\mathbf{T}(x) = \hat{\mathbf{T}}(x, \nabla\boldsymbol{\varphi}(x)), \quad \text{or equivalently} \\ &T_{ij}(x) = \hat{T}_{ij}(x, \nabla\boldsymbol{\varphi}(x)), \qquad 1 \le i, j \le 3, \end{aligned} \tag{7}$$

where the *response function*

$$\hat{\mathbf{T}} = (\hat{T}_{ij}) : (x, \mathbf{F}) \in \overline{\Omega} \times \mathbf{M}^3_+ \to \hat{\mathbf{T}}(x, \mathbf{F}) \in \mathbf{M}^3 \tag{8}$$

characterizes the elastic material. Note that, by (4), the response function must also satisfy

$$\hat{\mathbf{T}}(x, \mathbf{F})\mathbf{F}^T \in \mathbf{S}^3 \qquad \text{for all } (x, \mathbf{F}) \in \overline{\Omega} \times \mathbf{M}^3_+. \tag{9}$$

One must then take into account the *axiom of material frame-indifference*, a general principle in physics that, loosely speaking, asserts that any observable quantity with an "intrinsic" character (here, the Cauchy stress vector) must be independent of the particular basis in which it is computed. As expected, the effect of this axiom, also known as the *axiom of invariance under a change of observer*, or the *axiom of objectivity*, is to reduce the class of mappings of the form (8), and which satisfy (9), that may be used in a constitutive equation (7) of an elastic material.

More specifically, let $\mathbf{O}^3_+$ denote the set of all rotations in $\mathbf{R}^3$, i.e., orthogonal matrices $\mathbf{Q}$ of order 3 with $\det \mathbf{Q} = +1$. Then an elastic material is frame-indifferent if and only if, at each $x \in \overline{\Omega}$,

$$\hat{\mathbf{T}}(x, \mathbf{QF}) = \mathbf{Q}\hat{\mathbf{T}}(x, \mathbf{F}) \qquad \text{for all } \mathbf{Q} \in \mathbf{O}^3_+, \mathbf{F} \in \mathbf{M}^3_+. \tag{10}$$

As shown by Fosdick and Serrin [13], a noteworthy consequence of relation (10) is that *the response function* $\hat{\mathbf{T}}$ *cannot be linear with respect to its argument* $\mathbf{F} \in \mathbf{M}^3_+$ if the reference configuration is a *natural state*, i.e. if the stress tensor vanishes when $\boldsymbol{\varphi} = \mathrm{id}$ (this is equivalent to saying that $\hat{\mathbf{T}}(x, \mathbf{I}) = \mathbf{0}$ for all $x \in \overline{\Omega}$). Note in passing that this observation definitely rules out *linear* partial differential equations (as in (13) below) as a possible model of elasticity!

Let us assume that the unknown deformation $\boldsymbol{\varphi}$ satisfies a *boundary condition of place*, of the form

$$\boldsymbol{\varphi}(x) = \boldsymbol{\varphi}_0(x), \qquad x \in \Gamma_0, \tag{11}$$

where $\boldsymbol{\varphi}_0 : \Gamma_0 \to \mathbf{R}^3$ is a given mapping, on the remaining portion $\Gamma_0 = \Gamma - \Gamma_1$ of the boundary of Ω. We recall (cf. (1)) that on

Γ_1, $\boldsymbol{\varphi}$ satisfies a *boundary condition of traction*, of the form

$$\mathbf{T}(x)\mathbf{n}(x) = \hat{\mathbf{g}}(x, \nabla\boldsymbol{\varphi}(x)), \qquad x \in \Gamma_1. \tag{12}$$

Note that the boundary conditions of place and traction far from exhaust all the situations occurring in practice, where in particular *unilateral* boundary conditions are quite common.

Assembling the various notions found so far, we are thus seeking a deformation $\boldsymbol{\varphi}: \overline{\Omega} \to \mathbf{R}^3$ that solves the following *boundary value problem of three-dimensional elasticity* (we recall that a deformation must be orientation-preserving in $\overline{\Omega}$ and injective on Ω):

$$\begin{cases} -\mathbf{div}\,\hat{\mathbf{T}}(x, \nabla\boldsymbol{\varphi}(x)) = \hat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)), & x \in \Omega, \\ \boldsymbol{\varphi}(x) = \boldsymbol{\varphi}_0(x), & x \in \Gamma_0, \\ \hat{\mathbf{T}}(x, \nabla\boldsymbol{\varphi}(x))\mathbf{n}(x) = \hat{\mathbf{g}}(x, \nabla\boldsymbol{\varphi}(x)), & x \in \Gamma_1. \end{cases} \tag{13}$$

This problem is called a *pure displacement problem* if $\Gamma_1 = \varnothing$, a *displacement-traction problem* if area $\Gamma_0 > 0$ and area $\Gamma_1 > 0$, and a *pure traction problem* if $\Gamma_0 = \varnothing$.

Assuming appropriate differentiability, we can write the equations in Ω found in (13) as

$$\begin{aligned} &-\sum_{j,k,l=1}^{3} \frac{\partial \hat{T}_{ij}}{\partial F_{kl}}(x, \nabla\boldsymbol{\varphi}(x)) \frac{\partial^2 \varphi_k}{\partial x_j \partial x_l}(x) \\ &-\sum_{j=1}^{3} \frac{\partial \hat{T}_{ij}}{\partial x_j}(x, \nabla\boldsymbol{\varphi}(x)) = \hat{f}_i(x, \boldsymbol{\varphi}(x)), \qquad 1 \le i \le 3. \end{aligned} \tag{14}$$

In the terminology of partial differential equations, such second-order equations, whose higher-order terms are nonlinear functions of $\nabla\boldsymbol{\varphi}(x)$, are labeled *quasilinear*, by contrast with *semilinear* equations, where the terms containing the partial derivatives of the highest order are linear.

Quasilinear partial differential equations are considerably harder to analyze than semilinear ones; this is one reason why so many difficulties are encountered in the mathematical analysis of three-dimensional elasticity.

§3. Hyperelastic materials

An elastic material is *hyperelastic* if there exists a *stored energy function* $\widehat{W}: \overline{\Omega} \times \mathbf{M}^3_+ \to \mathbf{R}$ such that

$$\hat{\mathbf{T}}(x, \mathbf{F}) = \frac{\partial \widehat{W}}{\partial \mathbf{F}}(x, \mathbf{F}) \qquad \text{for all } x \in \overline{\Omega},\ \mathbf{F} \in \mathbf{M}^3_+, \tag{15}$$

or, componentwise,

$$\hat{T}_{ij}(x, \mathbf{F}) = \frac{\partial \hat{W}}{\partial F_{ij}}(x, \mathbf{F}) \qquad \text{for all } x \in \overline{\Omega},\ \mathbf{F} \in \mathbf{M}^3_+,\ 1 \le i,\ j \le 3.$$

Applied body and surface forces are *conservative* if there exist *potentials* $\hat{F}: \hat{\Omega} \times \mathbf{R}^3 \to \mathbf{R}$ and $\hat{G}: \Gamma_1 \times \mathbf{R}^3 \times \mathbf{M}^3_+ \to \mathbf{R}$ such that, for any smooth enough vector fields $\boldsymbol{\theta}: \overline{\Omega} \to \mathbf{R}^3$ that satisfy $\boldsymbol{\theta} = \mathbf{0}$ on Γ_0,

$$\int_\Omega \hat{\mathbf{f}}(x, \boldsymbol{\varphi}(x)) \cdot \boldsymbol{\theta}(x)\, dx = F'(\boldsymbol{\varphi})\boldsymbol{\theta}, \qquad \text{with } F(\boldsymbol{\psi}) = \int_\Omega \hat{F}(x, \boldsymbol{\psi}(x))\, dx, \tag{16}$$

$$\int_{\Gamma_1} \hat{\mathbf{g}}(x, \nabla\boldsymbol{\varphi}(x)) \cdot \boldsymbol{\theta}(x)\, dx = G'(\boldsymbol{\varphi})\boldsymbol{\theta}, \qquad \text{with } G(\boldsymbol{\psi}) = \int_{\Gamma_1} \hat{G}(x, \boldsymbol{\psi}(x), \nabla\boldsymbol{\psi}(x))\, dx, \tag{17}$$

where $\cdot$ denotes the Euclidean inner product in $\mathbf{R}^3$, and $F'(\boldsymbol{\varphi})$ and $G'(\boldsymbol{\varphi})$ denote the Fréchet derivatives at $\boldsymbol{\varphi}$ of the functionals F and G. For instance, dead loads, or a pressure load, are conservative.

If the material is hyperelastic and if the applied forces are conservative, solving the boundary value problem (13) is formally equivalent to finding the *stationary points* of the *total energy* I, defined by

$$I(\boldsymbol{\psi}) = \int_\Omega \hat{W}(x, \nabla\boldsymbol{\psi}(x))\, dx - \{F(\boldsymbol{\psi}) + G(\boldsymbol{\psi})\}, \tag{18}$$

when $\boldsymbol{\psi}$ varies in a *set of admissible deformations* of the form

$$\boldsymbol{\Phi} = \{\boldsymbol{\psi}: \overline{\Omega} \to \mathbf{R}^3;\ \boldsymbol{\psi} \text{ is injective on } \Omega, \det \nabla\boldsymbol{\psi} > 0 \text{ in } \overline{\Omega},\ \boldsymbol{\psi} = \boldsymbol{\varphi}_0 \text{ on } \Gamma_0\}. \tag{19}$$

In other words, the boundary value problem (13) forms the *Euler–Lagrange equations* associated with the total energy; in particular, any *minimizer* $\boldsymbol{\varphi}$ of the functional I over the set $\boldsymbol{\Phi}$, i.e., any $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ that satisfies

$$I(\boldsymbol{\varphi}) = \inf_{\boldsymbol{\psi} \in \boldsymbol{\Phi}} I(\boldsymbol{\psi}), \tag{20}$$

is a solution of this boundary value problem, provided it is smooth enough (note that a minimizer is a particular stationary point).