ENDS OF RIEMANNIAN MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE OUTSIDE A COMPACT SET

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ABSTRACT. We consider complete manifolds with Ricci curvature nonnegative outside a compact set and prove that the number of ends of such a manifold is finite and in particular, we give an explicit upper bound for the number.

1. INTRODUCTION

Toponogov [T] showed that in a complete manifold of non-negative sectional curvature, a line splits off isometrically, i.e. any nonnegatively curved \( M^n \) is isometric to a Riemannian product \( N^k \times R^{n-k} \), where \( N^k \) does not contain a line. Later, Cheeger and Gromoll [CG] generalized this to manifolds of nonnegative Ricci curvature, known as the Cheeger-Gromoll splitting theorem. As a consequence, such a manifold has at most two ends (see §2 for the definition of an end). In [A], Abresch studied manifolds with asymptotically nonnegative sectional curvature. He showed that the number of ends of such a manifold is finite and can be estimated from above explicitly. In this note, we consider manifolds with Ricci curvature being nonnegative outside a compact set and prove that the number of ends of such a manifold is finite and in particular, we give an explicit upper bound for the number. That is, we prove the following theorem.

Theorem. Let \((M^n, o)\) be a Riemannian manifold with base point \( o \). If the Ricci curvature is nonnegative outside the geodesic ball \( B(o, a) \) of radius \( a \) and is bounded from below on \( B(o, a) \) by \(- (n-1) \Lambda^2\) (for \( \Lambda \geq 0 \)), then there exists a universal bound on the number of ends, e.g.

\[
\text{the number of ends of } M^n \leq \frac{2n}{n-1} (\Lambda a)^{-n} \exp\left(\frac{17(n-1)}{2} \Lambda a\right).
\]

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We learned that P. Li and L. F. Tam proved a similar theorem as an application of the theory of harmonic functions on a complete manifold. Our approach here is more geometrical. A previous version of the Theorem, under the additional condition of a lower bound on the sectional curvature, was proved by Z. Liu. After reading a preliminary version of our paper, Z. Liu informed us that he could also modify his proof, using ideas from this paper, to prove the same theorem as above (see [LT, L]).

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2. IDEA OF THE PROOF OF THE THEOREM

In what follows, we always let $M^n$ be a manifold as in the Theorem.

There are various (but equivalent) definitions of an end of a manifold (cf. [A]), for the sake of our argument, we use the following definition.

**Definition 2.1.** Two rays $\gamma_1$ and $\gamma_2$ starting at the base point $o$ are called cofinal if for any $r > 0$ and any $t \geq r$, $\gamma_1(t)$ and $\gamma_2(t)$ lie in the same component of $M - B(o, r)$. An equivalence class of cofinal rays is called an end of $M$. We will use $[\gamma]$ to denote the class of the ray $\gamma$.

The following proposition is a key to the proof of the theorem.

**Proposition 2.2.** Let $M^n$ be as in the theorem, $[\gamma_1]$ and $[\gamma_2]$ be two different ends of $M^n$, then $d(\gamma_1(4a), \gamma_2(4a)) > 2a$.

Proposition 2.2 will be proved in §3. Assuming it, we now give a proof of the theorem.

**Proof of the theorem.** Let $k$ be an integer and $\gamma_1, ..., \gamma_k$ be rays from the base point $o$ going to $k$ different ends. We need to bound $k$ from above. Consider the sphere $S(o, 4a)$ of radius $4a$. Let $\{p_j\}$ be a maximal set of points on $S(o, 4a)$ such that the balls $B(p_j, \frac{1}{2}a)$ are disjoint. Clearly, the balls $B(p_j, a)$ cover $S(o, 4a)$, and since the set $\{\gamma_i(4a), i = 1, ..., k\}$ is contained in $S(o, 4a)$, each $\gamma_i(4a)$ is contained in some $B(p_j, a)$. But each ball $B(p_j, a)$ contains at most one $\gamma_i(4a)$ by the Proposition 2.2,
and hence the number of balls is not less than \( k \). Thus it suffices to bound the number of balls \( B(p_j, \frac{1}{2}a) \).

Notice that
\[
B(p_j, \frac{1}{2}a) \subset B(o, \frac{9}{2}a) \subset B(p_j, \frac{17}{2}a).
\]

It follows from the Bishop-Gromov volume comparison theorem that
\[
\text{vol} B(p_j, \frac{17}{2}a) \leq \int_{0}^{\frac{17a}{2}} \sinh^{n-1} \Lambda t \, dt \text{vol} B(p_j, \frac{1}{2}a).
\]

Therefore, the number of balls \( B(p_j, \frac{1}{2}a) \) is no more than
\[
\frac{\int_{0}^{\frac{1}{2}a} \sinh^{n-1} \Lambda t \, dt}{\int_{0}^{\frac{1}{4}a} \sinh^{n-1} \Lambda t \, dt}.
\]

Since
\[
\int_{0}^{\frac{17a}{2}} \frac{\sinh^{n-1} \Lambda t \, dt}{\sinh^{n-1} \Lambda t \, dt} \leq \frac{2n}{n-1} \frac{17(n-1)}{2} \Lambda a \frac{\Lambda a^n}{(\Lambda a)^n},
\]

the theorem follows.

Remark 2.3. The bound for the number of ends given here is far from being sharp. An improved bound can be obtained from a more general volume comparison theorem which we can state as follows (for definitions involved, one is referred to [AG]):

**A volume comparison theorem.** Let \( M^n \) be an asymptotically non-negatively Ricci curved manifold. Then for any \( p \in M^n \) and for every \( 0 \leq r \leq R \),
\[
\frac{\text{vol} B(p, R)}{\text{vol} B(p, r)} \leq w_n \left( \frac{R}{r} \right)^n
\]
where \( w_n = (1 + 2u(0)d(o, p))^{n-1} 2^n \exp(6(n-1)C_1) \).

Moreover, if \( 0 \leq r \leq R \leq d(o, p) \) or \( 2d(o, p) \leq r \leq R \), \( w_n \) can be chosen as \( 2^n \exp(6(n-1)C_1) \) (see [AG] for the definitions of \( u(0) \) and \( C_1 \)).

The proof of this theorem will appear elsewhere.

Proof of Proposition 2.2. Let \( M \) be a manifold as in the theorem.
For each ray \( \gamma \), there is an associated function called the Busemann function, which is defined as follows:

\[
b_\gamma(x) = \lim_{t \to \infty} (t - d(x, \gamma(t))).
\]

For any given point \( p \), let \( \alpha_t \) be a minimizing geodesic from \( p \) to \( \gamma(t) \). As \( t \to \infty \), \( \alpha_t \) has a convergent subsequence which converges to a ray at \( p \). Such a ray is called an asymptotic ray to \( \gamma \) at \( p \).

Let \( \gamma \) be a line. We define \( \gamma^+ : [0, \infty] \to M \) by \( \gamma^+(t) = \gamma(t) \) and \( \gamma^- : [0, \infty] \to M \) by \( \gamma^-(t) = \gamma(-t) \).

Let \( b_\gamma^+ \) (resp. \( b_\gamma^- \)) be the associated Busemann function of \( \gamma^+ \) (resp. \( \gamma^- \)).

In \([EH]\), J. Eschenburg and E. Heintze showed, under the assumption that the Ricci curvature is nonnegative everywhere, that \( b_\gamma^+ \) (resp. \( b_\gamma^- \)) are smooth harmonic functions with \( \text{Hess } b_\gamma^+ = 0 \) and \( b_\gamma^+ + b_\gamma^- = 0 \). Applying their arguments locally, we can show the following lemma.

**Lemma 3.1.** Let \( N \) be the \( \delta \)-tubular neighborhood of \( \gamma \). Suppose that from every point \( p \) in \( N \), there is an asymptotic ray to \( \gamma^+ \) and an asymptotic ray to \( \gamma^- \) such that the Ricci curvature is nonnegative on both asymptotic rays. Then through every point in \( N \), there is a line \( \alpha \) which, when parametrized properly, satisfies

\[
b_\gamma^+(\alpha^+(t)) = t \quad \text{and} \quad b_\gamma^-(\alpha^-(t)) = t.
\]

**Proof.** Let \( p \) be any point in \( N \). Applying arguments as in the proof of Lemma 3 in \([EH]\), we find that at \( p \), \( b_\gamma^+ + b_\gamma^- = 0 \), and \( b_\gamma^\pm \) are \( C^1 \) smooth with \( \| \text{grad } b_\gamma^\pm \| = 1 \). Hence the asymptotes to \( \gamma^\pm \) are uniquely determined at \( p \) and fit together to a line, say, \( \gamma_p \). Arguments as in the proof of Lemma 2 together with the concluding remarks in \([EH]\) imply that \( b_\gamma^+ \) (resp. \( b_\gamma^- \)) is actually \( C^\infty \) smooth with \( \text{Hess } b_\gamma^\pm = 0 \) on \( \gamma_p \). Thus the restriction of \( b_\gamma^\pm \) to \( \gamma_p \) must be a linear function with derivative 1. After a reparametrization of \( \gamma_p \), Lemma 3.1 then follows.

**Remark 3.2.** The same argument as in \([EH]\) of course also implies a local splitting for the metric in \( N \), under the assumptions of Lemma 3.1.

**Lemma 3.3.** \( M^n \) cannot admit a line \( \gamma \) with the following property:

(I) \[d(\gamma(t), B(o, a)) \geq |t| + 2a \quad \text{for all } t.\]
Proof. Suppose there were such a line $\gamma$. Consider the $a$-tubular neighborhood of $\gamma$. We claim that from any point $p$ in this neighborhood, all its asymptotic rays to $\gamma^+$ (or $\gamma^-$) are away from $B(o, a)$, in particular, the Ricci curvature is nonnegative on such a ray. In fact, let $s$ be such that $d(p, \gamma(s)) < a$, then,

$$d(p, \gamma^\pm(t)) \leq d(p, \gamma(s)) + d(\gamma(s), \gamma^\pm(t))$$

$$\leq a + |s| + t$$

but any curve from $p$ to $\gamma^\pm(t)$ passing through $B(o, a)$ has length

$$l \geq d(p, B(o, a)) + d(\gamma^\pm(t), B(o, a))$$

$$\geq d(\gamma(s), B(o, a)) + d(\gamma(\pm t), B(o, a)) - a$$

the last inequality follows from the property (I). Clearly, this implies that any minimizing geodesic, say, $\alpha_t$, from $p$ to $\gamma^\pm(t)$ does not pass through $B(o, a)$. Hence any convergent subsequence of $\alpha_t$ will converge to a ray which is away from $B(o, a)$. This proves the claim.

Next, we claim that through every point of the $a$-tubular neighborhood of $\gamma$, there exists a line with the property (I). Indeed, it follows from the above claim and Lemma 3.1 that through every point of the $a$-tubular neighborhood of $\gamma$, there is a line $\beta$ such that

$$b^+(\beta^+(t)) = t \quad \text{and} \quad b_\gamma^-(\beta^-(t)) = t.$$ 

We need to show that $\beta$ also has the property (I), i.e.

$$d(\beta(t), B(o, a)) \geq |t| + 2a \quad \text{for all} \quad t.$$

By symmetry, we may assume that $t \geq 0$. Then for any $r \geq 0$,

$$d(\beta(t), B(o, a)) \geq d(\gamma(r), B(o, a)) - d(\beta(t), \gamma(r))$$

$$\geq r - d(\beta(t), \gamma(r)) + 2a$$

(here we used the property (I) for $\gamma$). Letting $r \to \infty$ in the above inequality, we have

$$d(\beta(t), B(o, a)) \geq b^+_\gamma(\beta(t)) + 2a = t + 2a.$$ 

Now let $\alpha(t) : [0, d] \to M$ be a minimizing geodesic from $\gamma(0)$ to $o$, then there is a partition of the interval $[0, d]$:\n$$t_0 = 0 < t_1 < \cdots < t_k = d$$ such that $d(\alpha(t_i), \alpha(t_{i+1})) < a$. 

The last claim implies that there is a line through $\alpha(t_1)$ with the property (I). Continuing this process inductively, we would find a line with the property (I) through $\alpha(t_k)$, the base point $o$, which is absurd.

We are now in the position to prove Proposition 2.2.

**Proof of Proposition 2.2.** Suppose the contrary. That is, $d(\gamma_1(4a), \gamma_2(4a)) \leq 2a$. Since $[\gamma_1]$ and $[\gamma_2]$ are different ends, there exists an $A > 4a$ such that $\gamma_1(t)$ and $\gamma_2(t)$ are in different unbounded components of $M - B(o, A)$ for all $t > A$. Let $C_t$ $(t > A)$ be a minimizing geodesic joining $\gamma_1(t)$ and $\gamma_2(t)$. Then $C_t$ must pass through $B(o, A)$. In addition, we claim that the middle point $m_t$ of $C_t$ is in the ball $B(o, 2A)$. As a matter of fact, let $p$ be a point in $C_t \cap B(o, A)$ and without loss of generality we may assume that $d(p, \gamma_1(t)) \leq d(p, \gamma_2(t))$, then

\[
d(o, m_t) \leq d(o, p) + d(p, m_t)
\]

\[
\leq A + \frac{1}{2} \rho_t - d(p, \gamma_1(t))
\]

\[
\leq A + \frac{1}{2} \rho_t - (t - A)
\]

where $\rho_t =$ the length of $C_t$. Notice that

\[
\rho_t = d(\gamma_1(t), \gamma_2(t))
\]

\[
\leq d(\gamma_1(t), \gamma_1(4a)) + d(\gamma_1(4a), \gamma_2(4a)) + d(\gamma_2(4a), \gamma_2(t))
\]

\[
\leq 2(t - 4a) + 2a = 2t - 6a.
\]

Hence,

\[
d(o, m_t) \leq A + \frac{1}{2}(2t - 6a) - (t - A)
\]

\[
= 2A - 3a.
\]

This shows that $m_t$ is in the ball $B(o, 2A)$.

Now we reparametrize $C_t$ by translating the origin and with abuse of notation we still denote it by $C_t$ such that

\[
C_t(-\frac{1}{2} \rho_t) = \gamma_1(t), \quad C_t(0) = m_t, \quad C_t(\frac{1}{2} \rho_t) = \gamma_2(t).
\]

We claim that $C_t(s)$ satisfies property (I) for $-\frac{1}{2} \rho_t \leq s \leq \frac{1}{2} \rho_t$.

In fact, for any $s$ (we may assume $s \geq 0$),

\[
d(C_t(s), B(o, a)) \geq d(C_t(\frac{1}{2} \rho_t), B(o, a)) - (\frac{1}{2} \rho_t - s)
\]

\[
\geq (t - a) - (t - 3a) + s
\]

\[
= s + 2a
\]

where we used the fact $\rho_t \leq 2t - 6a$. Since $C_t(0) \in B(o, 2A)$ for all $t \geq A$, when $t \to \infty$, a subsequence of $C_t$ converges to a line $\gamma(s)$ with the property (I) for all $s$. (Notice that $\rho_t \to \infty$, as $t \to \infty$). This is a contradiction by Lemma 3.3.
REFERENCES


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