Actually this element is well defined only modulo an inner automorphism of G, so that to obtain a well-defined function on $\mathscr{A}/\mathscr{G}$, Witten chooses an auxiliary representation V of G, and defines

$$\chi_W(\underline{k}) = \operatorname{Trace} \rho(g_{\underline{k}})$$

as the " W-measure" of the holonomy determined by the oriented knot $\underline{k}$. In this way every pair (G, W) defines a numerical invariant on knots by its expectation value $\langle \chi_W(\underline{k}) \rangle$ in the k-G theory, and Witten argues that these expectation values are related to the values of the Jones and Alexander polynomials at various roots of unity.

For instance, when $G = SU(2)$ and W is the representation of "level k," then according to Witten

$$\langle \chi_W(\underline{k}) \rangle = V_{\underline{k}}(q), \qquad q = e^{2\pi i/2+k},$$

and I hope that the urge of discovering what is behind this intriguing explicit formula will propel the reader unhesitatingly to the bookstore for a copy of Atiyah's, *The geometry and physics of knots.*

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Variational principles of topology. Multidimensional minimal surface theory, by Anatoliĭ T. Fomenko. Kluwer Academic Publishers, Dordrecht, Boston, and London, 1990, 374 pp., $133.00. ISBN 0-7923-0230-3

Anatoliĭ T. Fomenko is the most prominent mathematician in the Soviet Union working in higher dimensional minimal surface theory. The book under review on area minimizing surfaces is a revised translation of his earlier volume

Variational methods in topology (published in Russian in 1982) based on a course which he taught to undergraduate and graduate students at Moscow State University. In 1990 Fomenko published a two volume work, *Plateau's problem* [F1], as a general survey of problems of least area in contrast with the work being reviewed which focuses mainly on his own contributions. Fomenko is also an accomplished artist; some of his work was recently published by the Society [F2].

The study of least area problems in general dimensions is largely a development of twentieth century mathematics. Prior to about 1960 a k-dimensional area minimizing surface in an n-dimensional Riemannian manifold likely would have been either a submanifold or the image of a k-dimensional manifold under a reasonable mapping. In the first case surface area would have been Hausdorff's intrinsic k-dimensional measure (the usual surface area of a submanifold) while in the latter case, area would have been the integral of the Jacobian of the mapping. Except when $k = 2$ there were essentially no existence theorems for area minimizing surfaces, at least with any regularity.

The year 1960 is frequently taken as a turning point in the geometric calculus of variations, and especially in the study of least area, because of three seminal contributions.

First was the paper *Normal and integral currents* [FF] by H. Federer and W. H. Fleming; this paper was awarded one of the Society's 1987 Steele Prizes. The best introductory reference to the mathematics which grew from this start is the book *Geometric measure theory. A beginner's guide* [MF] by F. Morgan.

A second contribution was work of E. De Giorgi which included, in particular, an almost everywhere regularity theorem for area minimizing oriented hypersurfaces. Even though this theory is formally a mathematical subset of geometric measure theory, it has its own structure and, for the study of area minimizing hypersurfaces, is more readily accessible. The best introductory reference is the book *Minimal surfaces and functions of bounded variation* [GE] By E. Giusti (incorrectly spelled Ciusti in the volume under review).

The third contribution in 1960 was the paper *Solution of the Plateau Problem for m-dimensional surfaces of varying topological type* [R1] by E. R. Reifenberg which gave the first almost everywhere manifold structure for area minimizing surfaces of general codimension. Reifenberg, who was killed in a mountaineering accident in the Dolomites in the mid 1960s, was a student of A. S. Besicovitch in whose pioneering works lie many roots of present theories. Fomenko's work is largely set in an extension of Reifenberg's context.

In each of these approaches and contexts, area minimizing surfaces are sought among surfaces of varying topological type and singularity structure. Some such freedom is necessary in any general theory because not every boundary or homology class can be spanned by a singularity free minimal surface, as cobordism theory shows [AB]. There are, however, significant differences between these approaches. Within the geometric measure theory school, surfaces are often "integral currents," de Rham currents which are strongly approximatable by Lipschitz singular chains. These currents form a chain complex with the same homology groups as singular chains and have strong approximation and compactness properties; these properties guarantee the existence of surfaces representing integral homology classes and minimizing integrals of elliptic parametric integrands. Integrals, including area, being minimized are usually computed with densities equal to topological multiplicity. Area weighted with such densities is called "mass" while area without multiplicity is sometimes called "size"—both are lower semicontinuous under weak convergence of currents, although a mass bound typically is necessary in order that a minimizing sequence have a convergent subsequence. It is a theorem of the reviewer that any mass minimizing integral current is a minimal submanifold except for a possible singular set of codimension at least two [A3]; this is the best possible general result since holomorphic varieties in Kähler manifolds are mass minimizing. The singular sets of mass minimizing oriented hypersurfaces are of codimension at least seven [FH] according to earlier work of the reviewer, E. Bombieri, E. De Giorgi, H. Federer, W. H. Fleming, E. Giusti, and J. Simons.

For Reifenberg, a surface was a compact set, and its area equaled its spherical Hausdorff measure, analogous to size rather than mass. These surfaces span their boundaries or homology classes in the sense of Čech homology. Reifenberg obtained the first regularity results for area minimizing surfaces in general dimensions and codimensions [R2] in 1964 by showing in $\mathbf{R}^n$ that his surfaces were almost everywhere smooth minimal submanifolds. His surfaces typically have singularities of codimension one along which three sheets of surface meet at equal angles as in soap films. Because area is not counted with multiplicities, the branching singularities of mass minimizing integral currents (as in holomorphic varieties) do not occur in his surfaces. Since Fomenko works in Reifenberg's context, when he says a minimal surface is the image of a manifold, the area in question is the Hausdorff measure of the image and not the Jacobian mapping area mentioned above.

There are several other types of measure theoretic surfaces in substantial use today. Flat chains $\bmod \nu$ (introduced in [ZW] in 1962 and in [FW] in 1966) are modelled on Lipschitz singular chains with coefficients in the integers modulo ν; an area minimizing Möbius band would thus be a flat chain $\bmod 2$. Varifold surfaces (introduced in [A1] in 1965 and in [AW] in 1972) are useful for the variational calculus in the large and motion by mean curvature [BK]; every compact Riemannian manifold, for example, contains a minimal hypersurface whose singular set has codimension at least seven [PJ]. The $(\mathbf{F}, \varepsilon, \delta)$ minimal surfaces (introduced in [A2] in 1976) are useful for constrained surface energy minimization such as occurs in soap bubble clusters; in mathematical models for soap films and soap bubble geometries as $(\mathbf{M}, \varepsilon, \delta)$ minimal sets, typically size is minimized and singular sets consist of smooth curves meeting at equal angles at isolated points according to J. Taylor [T1]. Multifunctions (introduced in [A3] in 1984 and [A4] in 1988) are useful for analysis of branching behavior of minimal surfaces [CS] and in the study of rectifiable currents with real densities.

Fomenko's book touches on a large number of topics in minimal surface theory; the table of contents is seven pages long! He lists three principal themes:

(1) *Stratified surfaces and stratified volumes.* One of Fomenko's main original contributions to the multivariable variational calculus is his introduction of interesting new constraints for area minimization called spectral homology and cohomology. This constitutes the central topic of the book. Roughly speaking, what is required for such a constraint is preservation of nontriviality under convergence of compact sets in the Hausdorff distance topology. Since the smallest

set carrying the nontriviality can be of lower dimension (the nontriviality of the normal bundle of a Möbius band in $\mathbf{R}^3$, for example, is carried by a curve) one is led to construct strata of minimal surfaces, each minimizing area in its own dimension subject to the condition that all higher-dimensional pieces already minimize their area. Fomenko proves the existence and almost everywhere regularity of each stratum by an extension of Reifenberg's arguments; regularity alternatively follows from the reviewer's paper [A2].

(2) *A new method for the proof of global minimality and new examples.* Some of Fomenko's especially nice contributions are examples of absolutely area minimizing surfaces in Lie groups and an apparent complete classification of locally minimal, totally geodesic submanifolds realizing nontrivial cycles and elements of homotopy groups in symmetric spaces.

(3) *A relation between Bott periodicity, topologically nontrivial, totally geodesic surfaces in Lie groups, and the number of independent vector fields on spheres.*

The reviewer has known Fomenko personally for more than two decades and still is at a loss to understand why he is not more responsible in his mathematical claims. The following are two particular examples of concern.

The book cover states "In this volume, the solution of the Plateau problem in the class of all manifolds with fixed boundary is given in detail ... " Fomenko made a similar claim in a lecture at and in the proceedings of the 1974 International Congress in Vancouver, in the introduction to a major paper (in Russian), and in an interview published in the Mathematical Intelligencer. His preface in the volume under review is ambiguous about this issue. In any case, the claim is not proved, as he acknowledges privately. It is not known at present whether or not minimal surfaces of the type he studies are necessarily representable as continuous images of manifolds or even as continuous images of sets of finite topological complexity. The only significant contributions to this representation problem are due to B. White [W1, W2] who worked in a somewhat different mathematical context.

A second example occurs in §8 of Chapter 2, entitled *Solution of the problem of finding globally minimal surfaces in each homotopy class of multivarifolds.* Fomenko asserts "Đao Chông Thi solved Plateau's problem by establishing the existence of a locally Lipschitz mapping $g_0: W^k \to M^n$ in terms of currents, which minimizes the k-dimensional volume functional in the class of all locally Lipschitz mappings $g: W \to M$ such that $g|_{\partial W} = f|_{\partial W}$ (the problem of finding the absolute minimum with respect to all homotopy classes of multivarifolds)." In fact, Thi did not prove such a theorem in papers known to the reviewer since he did not establish a common Lipschitz constant to his sequence of mappings. Both the reviewer and others have pointed this out to Fomenko in person. Yet he again makes his claim!

At this present time, the geometric calculus of variations is thriving theoretically and computationally. An overview of the theory as of 1984 can be found in the table of contents of the volume *Geometric measure theory and the calculus of variations* [AA]. A sampling of current developments such as mean curvature evolution and the crystalline variational calculus appears in the video tape and proceedings, *Computing optimal geometries* [T2]. The reviewer's paper *Questions and answers about area minimizing surfaces and geometric measure theory*

[A5] is also intended as an overview of where we are and where we would like to go.

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Induced modules over group algebras, by Gregory Karpilovsky. North-Holland Mathematics Studies, vol. 161, North-Holland, Amsterdam, 1990, 520 pp., $120.50. ISBN 0-444-88414-9

Frobenius introduced the idea of induced characters as a method of obtaining complex characters of a finite group from those of a subgroup. Frobenius Reciprocity shows that induction is in some sense dual to restriction of characters. Induction then became one of the main techniques in character theory, culminating in Brauer's Induction Theorem, which states that a character of a finite group is an integral linear combination of characters induced from subgroups that are the direct products of p-groups and cyclic groups. This has numerous important consequences; to name just two, Brauer showed that Artin L-functions are meromorphic, and that a splitting field for a group of exponent n can be obtained by adjoining an nth root of unity of the rationals.

In the 1950s D. G. Higman introduced the idea of an induced module. If H is a subgroup of a finite group G and M is a module for the group ring RH where R is any ring, then the induced module denoted M^G is defined as $RG \otimes_{RH} M$. Frobenius's theory can be recovered by taking R to be complex numbers. If R is a field of characteristic $p > 0$ there are numerous difficulties and open questions in representation theory, and most of this book is concerned with induced RG-modules where R is such a field or a p-adic ring (the integral closure of the p-adic integers $\mathbf{Z}_p$ in a finite extension field of the p-adic rationals.)

Some of the most useful work is due to J. A. Green. Since RG-modules in general need not be completely reducible, indecomposability becomes an important issue. Green's Indecomposability Theorem states that an absolutely indecomposable RH-module remains absolutely indecomposable upon induction to G if H is a normal subgroup of G of index p; here a module is termed absolutely indecomposable if it remains indecomposable upon extension of scalars R to a bigger field of characteristic p or a bigger p-adic ring. Green also introduced the useful concepts of vertices and sources. The vertex of an indecomposable RG-module is a certain p-subgroup of G that turns out to be defined up to conjugacy in G. There is a one-to-one correspondence, called the Green Correspondence, between isomorphism types of indecomposable RG-modules of vertex V and isomorphism types of indecomposable modules for the normalizer of V in G with vertex V. The theory has come full circle, as Alperin [1] has used the Green Correspondence to prove Brauer's Induction Theorem.