

SOME NON-ANALYTIC-HYPOELLIPTIC SUMS OF SQUARES OF VECTOR FIELDS

MICHAEL CHRIST

ABSTRACT. Certain second-order partial differential operators, which are expressed as sums of squares of real-analytic vector fields in \mathbb{R}^3 and which are well known to be C^∞ hypoelliptic, fail to be analytic hypoelliptic.

1. INTRODUCTION

A differential operator L is said to be analytic hypoelliptic if whenever u is a distribution such that Lu is real-analytic in some open set U , then u is necessarily also real-analytic in U . Elliptic operators with analytic coefficients are analytic hypoelliptic, as are certain classes of subelliptic operators [GS, M2, S, Ta, Tp, Tv1, Tv2]. It has been known for some time that many subelliptic operators—whose solutions are necessarily C^∞ —nonetheless fail to be analytic hypoelliptic; among the examples now known are [BG, M1, He, PR, HH, CG]. A substantial no-man’s-land persists, in which neither alternative has been proved, even in rather simple cases. In this note are announced negative results for certain second-order operators. We hope that these will serve as models for larger classes of operators, rather than being mere isolated examples.

In \mathbb{R}^3 with coordinates x, y, t set

$$(0) \quad X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y} - mx^{m-1} \frac{\partial}{\partial t},$$

and

$$L = X^2 + Y^2,$$

where $m \geq 2$ is an integer. Then L is hypoelliptic in the C^∞ sense [H1, K]; when $m = 2$, it is analytic hypoelliptic [M2, Ta, Tv2]. For $m \geq 3$ an odd integer, however, it is not analytic hypoelliptic. This was proved for $m = 3$ in [He, PR], and extended to larger m in [HH], but by a method which does not apply for m even. In [CG] it was found that $\bar{\partial}_b \circ \bar{\partial}_b^*$ fails to be microlocally analytic hypoelliptic in the appropriate part of phase space, on the CR manifold $\{\mathfrak{S}(z_2) = [\Re(z_1)]^m\}$, where $m \geq 4$ is even. In appropriate coordinates for this manifold, $-\bar{\partial}_b \circ \bar{\partial}_b^* = (X + iY) \circ (X - iY) = X^2 + Y^2 - i[X, Y]$, where X, Y are as in (0).

Theorem 1. *For any even integer $m \geq 4$, L is not analytic hypoelliptic.*

Despite the similarity to results just cited, this does not follow from previous methods. The proof is rooted in a phenomenon discovered for $\bar{\partial}_b \circ \bar{\partial}_b^*$ in [CG],

Received by the editors February 13, 1991 and, in revised form, June 10, 1991.

1991 *Mathematics Subject Classification.* Primary 35A20, 35B65, 35H05.

Research supported in part by grants from the National Science Foundation.

but that argument relied heavily upon an explicit formula for the Szegő kernel [N], for which there appears to be no analogue in the present situation.

To place Theorem 1 in context, consider two real vector fields X, Y in \mathbb{R}^3 with analytic coefficients, and suppose them to be linearly independent at each point. Say that a point $a \in \mathbb{R}^3$ is of type 2 if $X, Y, [X, Y]$ span the tangent space to \mathbb{R}^3 at a . A general result [Tv2, Ta, M2] guarantees analytic hypoellipticity at any point of type 2, leaving open the question of what sort of degeneracy is permitted. The following conjecture has been suggested in a more general form by Trèves [Tv2]: $L = X^2 + Y^2$ fails to be analytic hypoelliptic at a if in any neighborhood of a there exists a real curve γ , with $\gamma'(0) \neq 0$, such that

- $\gamma(t)$ is not a point of type 2 for any $|t| < \varepsilon$, and
- $\gamma'(t)$ belongs to the span of $X(\gamma(t)), Y(\gamma(t))$ for every $|t| < \varepsilon$.

One may hope that analytic hypoellipticity holds in all other cases. In the special case of Theorem 1, the plane $x = 0$ is foliated by a one-parameter family of such curves γ .

More recently we have built on the analysis outlined below to prove that analytic hypoellipticity breaks down for $X^2 + Y^2$, with $X = \partial_x$ and $Y = \partial_y - b'(x)\partial_t$, whenever b vanishes to order exactly m at some point, with $m \in \{3, 4, 5, \dots\}$.

2. OUTLINE OF PROOF

Let ζ, τ be variables dual to y, t . Taking a partial Fourier transform in these variables reduces the analysis of L to that of a two-parameter family of ordinary differential operators:

$$-\frac{d^2}{dx^2} + (\zeta - \tau mx^{m-1})^2.$$

A simple change of variables reduces the general case $\tau \neq 0$ to $\tau = 1$, so we set

$$\mathcal{L}_\zeta = -\frac{d^2}{dx^2} + (\zeta - mx^{m-1})^2.$$

It is well known [H2] that in order to prove that L is not analytic hypoelliptic, it suffices to demonstrate the next result (which is already known [PR, HH] for odd m).

Theorem 2. *Let $m \geq 3$ be an integer. Then there exist $\zeta \in \mathbb{C}$ and $f \in L^\infty(\mathbb{R})$, not identically equal to zero, satisfying $\mathcal{L}_\zeta f \equiv 0$.*

For then, assuming that ζ has strictly positive imaginary part, one may set

$$F(x, y, t) = \int_1^\infty e^{i\tau t + i\tau^{1/m}\zeta y} f(\tau^{1/m}x) d\tau$$

in the region $y > 0$. Then $F \in C^\infty$, and $LF \equiv 0$. If $f(0) \neq 0$, one calculates readily, via a change of the contour of integration, that

$$\left| \frac{\partial^k}{\partial t^k} F(0, 1, 0) \right| \geq \delta^{k+1}(mk)!$$

for some $\delta > 0$. Thus F is not real-analytic. If $f(0)$ does vanish, then $\frac{d}{dx}f(0) \neq 0$ and essentially the same reasoning applies to $\frac{\partial}{\partial x} \frac{\partial^k}{\partial t^k} F$. It is easy

to check that \mathcal{L}_ζ has a strictly positive lowest eigenvalue for each $\zeta \in \mathbb{R}$, and that for any ζ satisfying the conclusion of Theorem 2, $\bar{\zeta}$ does also; so the assumption above is legitimate.

We have only an indirect proof of the existence of (infinitely many) ζ with the property desired. Set $\gamma = -(m - 1)/2$, and $\Phi_\zeta(x) = \zeta x - x^m$. Since the coefficient of the first-order part of \mathcal{L}_ζ is zero, the Wronskian of any two solutions of \mathcal{L}_ζ is a constant function of x . In the next lemma we will have two such solutions for each ζ , so their Wronskian will be a function of ζ alone.

Lemma 3. *Let $m \geq 4$ be an even integer. For each $\zeta \in \mathbb{C}$ there exist functions f_ζ^+ and f_ζ^- defined on \mathbb{R} which satisfy $\mathcal{L}_\zeta f_\zeta^\pm \equiv 0$ and*

$$(1) \quad \left| f_\zeta^\pm(x) - e^{\Phi_\zeta(x)} |x|^\gamma \right| = O(|e^{\Phi_\zeta(x)}| \cdot |x|^{\gamma-1}) \quad \text{as } x \rightarrow \pm\infty,$$

respectively. These functions are unique, and depend holomorphically on ζ . Their Wronskian, W , satisfies

$$(2) \quad |W(\zeta)| \leq C \exp(C|\zeta|^{m/(m-1)}) \quad \forall \zeta \in \mathbb{C}$$

for some finite C and

$$(3) \quad |W(\zeta)| \geq \delta \exp(\delta|\zeta|^{m/(m-1)}) \quad \forall \zeta \in \mathbb{R},$$

for some $\delta > 0$.

Now, W must have at least one zero. If not, then the real part of $\log W$ would be a harmonic function on \mathbb{C}^1 with polynomial growth at infinity, hence would be a polynomial. By (2) and (3), its degree would have to be $m/(m - 1)$. But for $m \geq 3$, $m/(m - 1)$ is not an integer.¹

If $W(\zeta) = 0$, then f_ζ^- is a constant multiple of f_ζ^+ . Hence both decay exponentially as $x \rightarrow \pm\infty$, therefore certainly remain bounded. Thus f_ζ^+ is the function sought.

The same reasoning can be made to apply for odd $m \geq 3$, with a suitable modification of (1). Further argument shows that for any $\alpha \in \mathbb{R}$, the operator $X^2 + Y^2 + i\alpha[X, Y]$ fails to be analytic hypoelliptic. Related results appear in [C1, C2, C4].

The proof of Lemma 3 is entirely elementary; details will appear elsewhere [C3]. The existence of solutions f_ζ^\pm with the prescribed asymptotics is a special case of a standard result in the theory of ordinary differential equations with irregular singular points at infinity [CL].

REFERENCES

[BG] M. S. Baouendi and C. Goulaouic, *Nonanalytic-hypoellipticity for some degenerate elliptic operators*, Bull. Amer. Math. Soc. **78** (1972), 483–486.
 [C1] M. Christ, *Analytic hypoellipticity breaks down for weakly pseudoconvex Reinhardt domains*, Internat. Math. Res. Notices **1** (1991), 31–40.
 [C2] ———, *Remarks on the breakdown of analyticity for δ_b and Szegő kernels*, Proc. 1990 Sendai Conf. on Harmonic Analysis, Lecture Notes in Math. (to appear).
 [C3] ———, *Certain sums of squares of vector fields fail to be analytic hypoelliptic*, Comm. Partial Differential Equations (to appear).

¹Alternatively, the Hadamard product formula guarantees that any entire function of nonintegral order has infinitely many zeros.

- [C4] ———, *On the $\bar{\partial}$ equation in weighted L^2 norms in \mathbb{C}^1* , J. Geom. Anal. (to appear).
- [CG] M. Christ and D. Geller, *Counterexamples to analytic hypoellipticity for domains of finite type*, Ann. of Math. (to appear).
- [CL] E. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.
- [DZ] M. Derridj and C. Zuily, *Régularité analytique et Gevrey pour des classes d'opérateurs elliptiques paraboliques dégénérés du second ordre*, Astérisque **2**, 3 (1973), 371–381.
- [GS] A. Grigis and J. Sjöstrand, *Front d'onde analytique et sommes de carrés de champs de vecteurs*, Duke Math. J. **52** (1985), 35–51.
- [HH] N. Hanges and A. A. Himonas, *Singular solutions for sums of squares of vector fields*, preprint.
- [He] B. Helffer, *Conditions nécessaires d'hypoanalyticité pour des opérateurs invariants à gauche homogènes sur un groupe nilpotent gradué*, J. Differential Equations **44** (1982), 460–481.
- [H1] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171.
- [H2] ———, *The analysis of linear partial differential operators. I*, Springer-Verlag, Berlin, 1983.
- [K] J. J. Kohn, *Boundary behavior of $\bar{\partial}$ on weakly pseudo-convex manifolds of dimension two*, J. Differential Geom. **6** (1971), 523–542.
- [M1] G. Métivier, *Une class d'opérateurs non hypoélliptiques analytiques*, Indiana Univ. Math. J. **29** (1980), 823–860.
- [M2] ———, *Analytic hypoellipticity for operators with multiple characteristics*, Comm. Partial Differential Equations **6** (1981), 1–90.
- [N] A. Nagel, *Vector fields and nonisotropic metrics*, Beijing Lectures in Harmonic Analysis, Ann. of Math. Stud., no. 112, Princeton Univ. Press, Princeton, NJ, 1986, pp. 241–306.
- [PR] Pham The Lai and D. Robert, *Sur un problème aux valeurs propres non linéaire*, Israel J. Math. **36** (1980), 169–186.
- [S] J. Sjöstrand, *Analytic wavefront sets and operators with multiple characteristics*, Hokkaido Math. J. **12** (1983), 392–433.
- [Ta] D. S. Tartakoff, *On the local real analyticity of solutions to \square_b and the $\bar{\partial}$ -Neumann problem*, Acta Math. **145** (1980), 117–204.
- [Tp] J.-M. Trepeau, *Sur l'hypoellipticité analytique microlocale des opérateurs de type principal*, Comm. Partial Differential Equations **9** (1984), 1119–1146.
- [Tv1] F. Trèves, *Analytic-hypoelliptic partial differential equations of principal type*, Comm. Pure Appl. Math. **24** (1971), 537–570.
- [Tv2] ———, *Analytic hypo-ellipticity of a class of pseudodifferential operators with double characteristics and applications to the $\bar{\partial}$ -Neumann problem*, Comm. Partial Differential Equations **3** (1978), 475–642.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024
 E-mail address: christ@math.ucla.edu