Let $G$ be a semisimple linear algebraic group defined over $\mathbb{Q}$. In the arithmetic theory of automorphic forms the lattice $\Gamma = G(\mathbb{Z})$ and its congruence subgroups
\[ \Gamma(N) = \{ \gamma \in G(\mathbb{Z}) : \gamma \equiv I(N) \} , \quad N \in \mathbb{N} \]
play a central role. A basic problem is to understand the decomposition into irreducibles of the regular representation of $G(\mathbb{R})$ on $L^2(\Gamma(N) \backslash G(\mathbb{R}))$. In general this representation will not be a direct sum of irreducibles, and for our purposes of defining the spectrum, it is best to use the notions of weak containment and Fell topology on the unitary dual $\widehat{G}(\mathbb{R})$ of the Lie group $G(\mathbb{R})$. (See [D, 18.1].) For any closed subgroup $H$ of $G(\mathbb{R})$ we define the spectrum $\sigma(H \backslash G(\mathbb{R}))$ to be the subset of $\widehat{G}(\mathbb{R})$ consisting of all $\pi \in \widehat{G}(\mathbb{R})$ that are weakly contained in $L^2(H \backslash G(\mathbb{R}))$. Furthermore, if $\widehat{G}^1(\mathbb{R})$ is the set of irreducible spherical representations, we set $\sigma(H \backslash G(\mathbb{R})) := \sigma(H \backslash G(\mathbb{R})) \cap \widehat{G}^1(\mathbb{R})$. When $H = \Gamma(N)$, $\sigma(\Gamma(N) \backslash G(\mathbb{R}))$ consists of all $\pi \in \widehat{G}(\mathbb{R})$ occurring as subrepresentations of $L^2(\Gamma(N) \backslash G(\mathbb{R}))$ as well as those $\pi$'s that are in wave packets of unitary Eisenstein series [La]. The latter occur only when $\Gamma \backslash G(\mathbb{R})$ is not compact. We now introduce the central object of this note.

**Definition.** The automorphic (resp. Ramanujan) dual of $G$ is defined by
\[
\widehat{G}_{\text{Aut}} = \bigcup_{N=1}^{\infty} \sigma(\Gamma(N) \backslash G(\mathbb{R})) ,
\]
\[
\widehat{G}_{\text{Raman}} = \widehat{G}_{\text{Aut}} \cap \widehat{G}^1(\mathbb{R}) .
\]
Here closure is taken in the topological space $\widehat{G}(\mathbb{R})$.

Thus, $\widehat{G}_{\text{Aut}}$ is the smallest closed set containing all the congruence spectrum. Here is an alternative description of $\widehat{G}_{\text{Raman}}$. Let $G(\mathbb{R}) = KAN$ be an Iwasawa decomposition of $G(\mathbb{R})$; then the theory of spherical functions identifies $\widehat{G}^1(\mathbb{R})$ with a subset of $A_+^1/W$, where $A = \text{Lie} A$, $W = \text{Weyl}(G, A)$. Moreover, the Fell topology on $\widehat{G}^1(\mathbb{R})$ coincides with the topology of $\widehat{G}^1(\mathbb{R})$ viewed as a subset of $A_+^1/W$. Let $D$ be the ring of invariant differential operators on the associated symmetric space $X$. Then the duality theorem [GPS] shows that the spectrum
of \(D\) in \(L^2(\Gamma\backslash X)\), say \(\text{Sp}_T(D) \subset \mathfrak{A}_L/W\) is the image of \(\sigma^1(\Gamma \backslash G(\mathbb{R}))\) in \(\mathfrak{A}_L/W\) under the above identification. In particular, \(\widehat{G}_{\text{Raman}}\) is identified with
\[
\bigcup_{N=1}^{\infty} \text{Sp}_T(N)(D) \subset \mathfrak{A}_L/W.
\]

That there should be restrictions on \(\widehat{G}_{\text{Raman}}\) and \(\widehat{G}_{\text{Aut}}\) has its roots in the representation theoretic reinterpretation of the classical Ramanujan conjectures due to Satake [Sa]. Identifying the above sets may be viewed as the general Ramanujan conjectures. For example, Selberg’s 1/4-conjecture may be stated as follows: For \(G = \text{SL}_2\),

\[
\widehat{G}_{\text{Raman}} = \{1\} \cup \widehat{G}^1(\mathbb{R})_{\text{temp}},
\]

where, in general, \(\widehat{G}(\mathbb{R})_{\text{temp}} := \sigma(G(\mathbb{R}))\) is the set of tempered representations, and \(\widehat{G}^1(\mathbb{R})_{\text{temp}} = \widehat{G}(\mathbb{R})_{\text{temp}} \cap \widehat{G}^1(\mathbb{R})\). (See [CHH] for equivalent definitions of temperedness.)

While the individual sets \(\sigma(\Gamma(N) \backslash G(\mathbb{R}))\) are intractable, the set \(\widehat{G}_{\text{Aut}}\) (and \(\widehat{G}_{\text{Raman}}\)) enjoy certain functorial properties.

**Theorem 1.** Let \(G\) be a connected semisimple linear algebraic group defined over \(\mathbb{Q}\) and \(H < G\) a \(\mathbb{Q}\)-subgroup

(i) \(\text{Ind}_{H(\mathbb{R})}^{G(\mathbb{R})} \widehat{H}_{\text{Aut}} \subset \widehat{G}_{\text{Aut}}\).

(ii) Assume that \(H\) is semisimple; then
\[
\text{Res}_{H(\mathbb{R})} \widehat{G}_{\text{Aut}} \subset \widehat{H}_{\text{Aut}}.
\]

(iii) \(\widehat{G}_{\text{Aut}} \otimes \widehat{G}_{\text{Aut}} \subset \widehat{G}_{\text{Aut}}\).

A word about the meaning of these inclusions. Firstly, \(\text{Ind}\) denotes unitary induction and \(\text{Res}\) stands for restriction. By the inclusion, say in (i), we mean that if \(\pi' \in \widehat{H}_{\text{Aut}}\) and \(\pi\) is weakly contained in \(\text{Ind}_{H(\mathbb{R})}^{G(\mathbb{R})} \pi'\) then \(\pi \in \widehat{G}_{\text{Aut}}\). (i) produces (after a local calculation) elements in \(\widehat{G}_{\text{Aut}}\) from ones in \(\widehat{H}_{\text{Aut}}\) and yields a new method for constructing automorphic representations. Observe also that if \(\pi \in \widehat{G}(\mathbb{R})\) is an isolated point then \(\pi \in \widehat{G}_{\text{Aut}}\) implies that \(\pi\) occurs as a subrepresentation in \(L^2(\Gamma(N) \backslash G(\mathbb{R}))\) for some \(N\). This fact will be used below to construct certain automorphic cohomological representations. (ii) allow one to transfer setwise upper bounds on \(\widehat{H}_{\text{Aut}}\) to \(\widehat{G}_{\text{Aut}}\) and for many \(G\)'s gives nontrivial approximations to the Ramanujan conjectures. (iii) exhibits a certain internal structure of the set \(\widehat{G}_{\text{Aut}}\). We illustrate the use of Theorem 1 with some examples.

**Example A.** If \(H = \{e\}\) then (i) implies that
\[
\widehat{G}_{\text{Aut}} \supset \widehat{G}(\mathbb{R})_{\text{temp}} \cup \{1\}.
\]
When \(G(\mathbb{Z})\) is cocompact this follows also from de George-Wallach [GW]. In comparison with (2) one might hope that (3) is an equality. However, using other \(H\)'s and (i) one finds typically that \(\widehat{G}_{\text{Aut}}\) contains nontrivial, nontempered spectrum. For \(G = \text{Sp}(4)\) the failure of the naive Ramanujan conjecture has been observed by Howe and Piatetski-Shapiro [HP-S] using theta liftings.
Example B. Let \( k/\mathbb{Q} \) be a totally real field, \( q \) a quadratic form over \( k \) such that \( q \) has signature \((n, 1)\) over \( \mathbb{R} \), and all other conjugates are definite. Let \( G = \text{Res}_{k/\mathbb{Q}} SO(q) \). Then \( G(\mathbb{R}) \) is of \( \mathbb{R} \)-rank one and the noncompact factor is \( SO(n, 1) \). We identify \( \mathbb{R}^* \) with \( \mathbb{R} \) by sending \( \rho \) to \((n - 1)/2\). With this normalization \( \hat{G}(\mathbb{R})^1 \) is identified with \( i\mathbb{R} \cup [-\rho, \rho] \subset \mathbb{C} \) modulo \( \{ \pm 1 \} \). [K]. We parametrize \( \hat{G}(\mathbb{R}) \) by \( s \in i\mathbb{R}^+ \cup [0, \rho] \) and denote the corresponding representation by \( \pi_s \). Let \( \varphi_0, \ldots, \varphi_n \) be an orthogonal basis of \( q \) such that \( q(\varphi_i) > 0 \), \( 0 \leq i \leq n - 1 \), and \( q(\varphi_n) < 0 \). Define \( H = \text{Res}_{k/\mathbb{Q}} \{ g \in SO(q) : g(\varphi_i) = \varphi_i \} \).

Applying Theorem 1(i) to the trivial representation \( 1 \in \hat{H}_{\text{Aut}} \) we find that
\[
\sigma(H(\mathbb{R}) \backslash G(\mathbb{R})) \subset \hat{G}_{\text{Aut}}.
\]
Now \( \sigma^1(H(\mathbb{R}) \backslash G(\mathbb{R})) \) has been computed ([F]), and we find
\[(4) \quad \hat{G}_{\text{Raman}} \supset \{ \rho, \rho - 1, \rho - 2, \ldots \} \cup i\mathbb{R}^+.
\]
In particular, for \( n \geq 4 \) there are nontrivial nontempered spherical automorphic representations.

To find upper bounds on \( \hat{G}_{\text{Raman}} \) one uses Theorem 1(ii) and
\[
H = \text{Res}_{k/\mathbb{Q}} \{ g \in SO(q) : g(\varphi_i) = \varphi_i, \ 1 \leq i \leq n - 4 \}.
\]
Combining the Jacquet-Langlands correspondence [JL] with the Gelbart-Jacquet lift [GJ] one concludes that
\[
\hat{H}_{\text{Raman}} \subset i\mathbb{R}^+ \cup [0, \frac{1}{2}] \cup \{ 1 \}.
\]
Applying (ii) it follows that
\[(5) \quad \hat{G}_{\text{Raman}} \subset i\mathbb{R}^+ \cup [0, \rho - \frac{1}{2}] \cup \{ \rho \}.
\]
In the special case \( k = \mathbb{Q}, n \geq 4 \) this result has also been obtained by [EGM] and [LP-SS] using Poincaré series. Assuming the Ramanujan conjecture at \( \infty \) for \( \text{GL}(2) \) one deduces
\[(6) \quad \hat{G}_{\text{Raman}} \subset i\mathbb{R}^+ \cup [0, \rho - 1] \cup \{ \rho \}.
\]
(Compare with (4).) The natural conjecture arising from (4) and (6) is
\[
\hat{G}_{\text{Raman}} = i\mathbb{R}^+ \cup \{ \rho, \rho - 1, \ldots \}.
\]
This is apparently consistent with Arthur’s conjectures [A].

Example C. Let \( F_{4(-20)} \) be the \( \mathbb{R} \)-rank one form of \( F_4 \). Using a method of Borel [B], one may find \( \mathbb{Q} \)-groups \( H < G \) such that \( G(\mathbb{R}) \), \( H(\mathbb{R}) \) both have rank one, the noncompact simple factors being \( F_{4(-20)} \) and \( \text{Spin}(8, 1) \) respectively. With notations similar to Example B, one may identify \( \hat{G}(\mathbb{R}) \) with \( i\mathbb{R}^+ \cup [0, 5] \cup \{ 11 \} \), here \( \rho = 11 \). One may compute \( \sigma^1(H(\mathbb{R}) \backslash G(\mathbb{R})) \) and using Theorem 1(i) find that
\[
\hat{G}_{\text{Raman}} \supset i\mathbb{R}^+ \cup \{ 3, 11 \}.
\]
Example D. Consider now \( F_{4(4)} \), the split real form of \( F_4 \). The corresponding symmetric space has dimension 28. For any cocompact lattice \( \Gamma \subset F_{4(4)} \) one knows from Vogan-Zuckerman [VZ] that the Betti numbers \( \beta^m(\Gamma) = 0 \) for \( 0 < m < 8 \) or \( 20 < m < 28 \), \( m \neq 4 \) or \( 24 \), in these latter dimensions all the
cohomology comes from parallel forms of the symmetric space. Nevertheless using Theorem 1(i) we have

**Theorem 2.** For any cocompact lattice $\Gamma$ in $F_{4(4)}$ and $N \geq 0$ there exists $\Gamma' \subseteq \Gamma$ of finite index such that $\beta^m(\Gamma')^H \geq N$ for $m = 8, 20$.

The proof of Theorem 2 makes use of Matsushima's formula [BW] together with a recent result of Vogan ensuring that the unitary representation contributing to the above Betti numbers is isolated in the unitary dual of $F_{4(4)}$. The $\mathbb{Q}$-subgroup that we use in applying Theorem 1(i) has real points equal to $\text{Spin}(5, 4)$ up to compact factors. By the well-known result of Oshima-Matsuki [MO], we conclude that the discrete series of the symmetric space $F_{4(4)}/\text{Spin}(5, 4)$ contain a unitary representation with nonzero cohomology in degrees 8 and 20, which is isolated in the unitary dual. The fact that we have dealt with every lattice in $F_{4(4)}$ follows from Margulis's arithmeticity theorem [M], together with the classification of algebraic groups over number fields [T].

This method of constructing cohomology is rather general. If $\pi$ is isolated in $\widehat{G}(\mathbb{R})$ and is contained in the automorphic dual of $G$ then it occurs discretely in $L^2(\Gamma \backslash G(\mathbb{R}))$ for $\Gamma$ a congruence subgroup of deep enough level. David Vogan has recently obtained the necessary and sufficient conditions for a unitary representation with nonzero cohomology to be isolated, which implies that most of them do. Theorem 1 then allows us to obtain nonvanishing of cohomology in a large number of cases.

To end, we remark that these ideas extend in a natural way to $S$-arithmetic groups. The proof of Theorem 1(i) consists of approximating, in a suitable way, congruence subgroups of $H(\mathbb{Z})$ by congruence subgroups of $G(\mathbb{Z})$ and then applying criteria of weak containment. For the proofs of Theorem 1(ii), (iii) we refer the reader to [BS].

**ACKNOWLEDGMENTS**

We would like to thank David Vogan for sharing his insights into unitary representations and F. Bien for interesting conversations.

**References**


**Graduate Center, City University of New York, New York 10036**

**Department of Mathematics, University of Maryland, College Park, Maryland 20742**

**Department of Mathematics, Princeton University, Princeton, New Jersey 08544 and IBM Research Division, Almaden research center, 650 Harry Rd, San José, California 95120**