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A THEORY OF ALGEBRAIC COCYCLES

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ABSTRACT. We introduce the notion of an algebraic cocycle as the algebraic analogue of a map to an Eilenberg-MacLane space. Using these cocycles we develop a "cohomology theory" for complex algebraic varieties. The theory is bigraded, functorial, and admits Gysin maps. It carries a natural cup product and a pairing to L-homology. Chern classes of algebraic bundles are defined in the theory. There is a natural transformation to (singular) integral cohomology theory that preserves cup products. Computations in special cases are carried out. On a smooth variety it is proved that there are algebraic cocycles in each algebraic rational (p, p)-cohomology class.

In this announcement we present the outlines of a cohomology theory for algebraic varieties based on a new concept of an algebraic cocycle. Details will appear in [FL]. Our cohomology is a companion to the L-homology theory recently studied in [L, F, L-F1, L-F2, FM]. This homology is a bigraded theory based directly on the structure of the space of algebraic cycles. It admits a natural transformation to integral homology that generalizes the usual map taking a cycle to its homology class. Our new cohomology theory is similarly bigraded and based on the structure of the space of algebraic cocycles. It carries a ring structure coming from the complex join (an elementary construction of projective geometry), and it admits a natural transformation Φ to integral cohomology. Chern classes are defined in the theory and transform under Φ to the usual ones. Our definition of cohomology is very far from a duality construction on L-homology. Nonetheless, there is a natural and geometrically defined Kronecker pairing between our "morphic cohomology" and L-homology.

The foundation stone of our theory is the notion of an effective algebraic cocycle, which is of some independent interest. Roughly speaking, such a cocycle on a variety X, with values in a projective variety Y, is a morphism from X to the space of cycles on Y. When X is normal, this is equivalent (by "graphing") to a cycle on $X \times Y$ with equidimensional fibres over X. Such cocycles abound in algebraic geometry and arise naturally in many circumstances. The simplest perhaps is that of a flat morphism $f: X \to Y$ whose corresponding cocycle associates to $x \in X$, the pullback cycle $f^{-1}(\{x\})$. Many more arise naturally from synthetic constructions in projective geometry. We show that every variety is rich in cocycles. Indeed if X is smooth and projective, then every rational cohomology class that is Poincaré dual to an algebraic cycle is represented by (i.e., is $\Phi \otimes \mathbb{Q}$ of) an algebraic cocycle.

In what follows the word *variety* will denote a reduced, irreducible, locally

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closed subscheme of some complex projective space. Such a variety is called projective if it is in fact closed (equivalently, compact) in some projective space.

Definition 1. Given a projective variety $Y \subset P^n$ with a fixed embedding, we denote by $C_d^s(Y)$ the algebraic set (the "Chow set") of effective cycles of codimension-s and degree d with support in Y. For an arbitrary variety X we then define an *effective algebraic cocycle* on X with values in Y to be a continuous algebraic map $\varphi: X \to C_d^s(Y)$ (i.e., a morphism $\varphi: \tilde{X} \to C_d^s(Y)$ from the weak normalization $\tilde{X}$ of X). The space of all such cocycles provided with the compact-open topology will be denoted by $C_d^s(X; Y)$.

Note that $C_d^s(X; Y)$ is *a priori* a hybrid construction consisting of algebrogeometric objects but carrying the compact-open topology. However, for normal projective varieties X, it has a purely algebraic description.

Proposition 2. *If X is projective and normal, then the space $C_d^s(X; Y)$ admits the structure of a locally closed, reduced subscheme of some complex projective space.*

Formal direct sum determines an abelian topological monoid structure on the spaces

$$C^s(X; Y) \stackrel{\text{def}}{=} \coprod_{d \geq 0} C_d^s(X; Y).$$

This structure is proved to be independent of the projective embedding chosen for Y. In analogy with the construction of L-homology, we have the following definition.

Definition 3. Let X and Y be varieties, with Y projective. Denote by $Z^s(X; Y)$ the homotopy theoretic group completion of $C^s(X; Y)$ (i.e., the loops on the classifying space of $C^s(X; Y)$),

$$Z^s(X; Y) \equiv \Omega B(C^s(X; Y)).$$

Then the *bivariant morphic cohomology* of X with coefficients in Y is defined to be the homotopy groups of $Z^s(X; Y)$,

$$L^s H^q(X; Y) \equiv \pi_{2s-q}(Z^s(X; Y)), \qquad 2s \geq q \geq 0.$$

The first fundamental result concerning these spaces is the Algebraic Suspension Theorem, which asserts that the algebraic suspension maps $C_d^s(X; Y) \to C_d^s(X; \not{\Sigma} Y)$, introduced in [L], induce a homotopy equivalence

$$Z^s(X; Y) \xrightarrow{\cong} Z^s(X; \not{\Sigma} Y),$$

and thus an isomorphism

$$L^s H^q(X; Y) \cong L^s H^q(X; \not{\Sigma} Y)$$

for all s and q.

Although this theory has been developed in the "bivariant context" of Definition 3, we shall focus our attention here on the important special case in which Y is some projective space P^N. We know from [L] that $\Omega B(C^s(P^N))$ has the homotopy type of a generalized Eilenberg-MacLane space

$$\Omega B(C^s(P^N)) \cong K(\mathbf{Z}, 0) \times K(\mathbf{Z}, 2) \times \cdots \times K(\mathbf{Z}, 2s), \qquad N \geq s.$$

In particular, this homotopy type is independent of $N \geq s$. We conclude that $\Omega B(C^s(P^N))$ "modulo" $\Omega B(C^{s-1}(P^{N-1}))$ represents cohomology. This motivates the following

Definition 4. For any variety X and any $s \geq 0$, let $Z^s(X)$ denote the homotopy fibre of the natural map $BC^{s-1}(X, P^{s-1}) \to BC^s(X, P^s)$,

$$Z^s(X) \equiv \text{htyfib}\{BC^{s-1}(X, P^{s-1}) \to BC^s(X, P^s)\}.$$

For any $0 \leq q \leq 2s$, the *morphic cohomology* group $L^sH^q(X)$ is defined by

$$L^sH^q(X) \equiv \pi_{2s-q}(Z^s(X)).$$

Definitions 3 and 4 are related as follows.

Theorem 5. *For any variety X and any $N \geq s \geq 0$, there is a natural homotopy equivalence*

$$Z^s(X; P^N) \cong Z^0(X) \times Z^1(X) \times \cdots \times Z^s(X).$$

The splitting asserted in Theorem 5 arises from natural maps

$$SP^\infty(P^N) \to SP^\infty(P^k), \qquad k \leq N$$

obtained by viewing P^N as $SP^N(P^1)$ and P^k as $SP^k(P^1)$, where $SP^j(P^n)$ denotes the j-fold symmetric product of P^n.

We view morphic cohomology as the theory corresponding to "algebraic" as opposed to "arbitrary continuous" maps from X into Eilenberg-MacLane spaces. This perspective is formalized in the following.

Theorem 6. *For any variety X, the elementary complex join operation induces a natural ring structure on*

$$L^{\bullet}H^{\bullet}(X) \equiv \bigoplus_{s\geq 0} L^sH^{\bullet}(X).$$

Furthermore, there is a natural transformation of graded rings

$$\Phi\colon L^{\bullet}H^{\bullet}(X) \to H^{\bullet}(X; \mathbf{Z}),$$

and, if X is projective then

$$\Phi(L^sH^{2s-j}(X)) \otimes \mathbf{C} \subset H^{s,s-j} \oplus H^{s-1,s-j+1} \oplus \cdots \oplus H^{s-j,j},$$

where $H^{p,q}$ denotes the (p,q)th Dolbeault component of $H^{p+q}(X; \mathbf{C})$.

The existence of a ring structure in morphic cohomology provides it with a structure not possessed by L-homology. On the other hand, the natural operations on L-homology constructed in [FM] via the join operation naturally determine operations on our morphic cohomology groups.

The restriction on the image of Φ given in Theorem 6 is complemented by the following existence result. A key ingredient in its proof is the total Chern class map of [LM],

$$BU_s \to C^s(P^\infty),$$

which can be viewed geometrically as the inclusion of degree 1 cycles into the space of all cycles on P^N for N sufficiently large.

Theorem 7. *Let E be a vector bundle over X generated by its global sections, over a variety X. Then there are naturally defined chern classes*

$$c_k(E) \in L^k H^{2k}(X)$$

with the property that $\Phi(c_k(E)) \in H^{2k}(X; \mathbf{Z})$ is the usual kth chern class of E. Consequently, if X is a smooth projective variety, then the Poincare dual of the fundamental class of each algebraic subvariety lies in the subring of $H^*(X; \mathbf{Z})$ generated $\Phi(L^\bullet H^\bullet(X))$.

Not surprisingly, codimension-1 morphic cohomology is the easiest to compute. We have the following computation.

Theorem 8. *Let X be a projective variety. Then*

$$L^1 H^q(X) = \begin{cases} \mathbf{Z} & \text{if } q = 0, \\ H^1(X; \mathbf{Z}) & \text{if } q = 1, \\ H^{1,1}(X; \mathbf{Z}) & \text{if } q = 2, \\ 0 & \text{otherwise}, \end{cases}$$

where $H^{1,1}(X; \mathbf{Z})$ denotes $H^2(X; \mathbf{Z}) \cap \rho^{-1} H^{1,1}(X; \mathbf{C})$ and where $\rho \colon H^2(X; \mathbf{Z}) \to H^2(X; \mathbf{C})$ is the coefficient homomorphism.

In [F] a result similar to Theorem 8 but only applying to smooth projective varieties was proved for L-homology. This suggests that morphic cohomology and L-homology should satsify some form of duality. One possible candidate for a possible duality pairing is given in the following proposition.

Proposition 9. *For any variety X, there is a natural Kronecker pairing between L-homology and morphic cohomology,*

$$L^s H^q(X) \otimes L_r H_q(X) \to \mathbf{Z},$$

which is naturally compatible with the usual Kronecker pairing

$$H^q(X; \mathbf{Z}) \otimes H_q(X; \mathbf{Z}) \to \mathbf{Z}.$$

The definition of this pairing is pleasingly geometric. Namely, given a cycle W on $X \times P^N$ and a cycle Z on X, we take the image in $H_*(P^N; \mathbf{Z})$ of the fundamental class of the restriction of W to Z.

References

[F] E. Friedlander, *Algebraic cycles, Chow varieties, and Lawson homology*, Compositio Math. 77 (1991), 55–93.

[FL] E. Friedlander and H. B. Lawson, *A theory of algebraic cocycles*, Ann. of Math. (to appear).

[FM] E. Friedlander and B. Mazur, *Filtrations on the homology of algebraic varieties* (to appear).

[L-F1] P. Lima-Filho, *On a homology theory for algebraic varieties*, IAS preprint, 1990.

[L-F2] ——, *Completions and fibrations for topological monoids and excision for Lawson homology*, Compositio Math., (1991).

[L] H. B. Lawson, Jr., *Algebraic cycles and homotopy theory*, Ann. of Math. (2) **129** (1989), 253–291.

[LM] H. B. Lawson, Jr. and M.-L. Michelsohn, *Algebraic cycles, Bott periodicity, and the Chern characteristic map*, The Mathematical Heritage of Herman Weyl, Amer. Math. Soc., Providence, RI, 1988, pp. 241–264.

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