

## A CLASSIFICATION OF THE STABLE TYPE OF $BG$

JOHN MARTINO AND STEWART PRIDDY

**ABSTRACT.** We give a classification of the  $p$ -local stable homotopy type of  $BG$ , where  $G$  is a finite group, in purely algebraic terms.  $BG$  is determined by conjugacy classes of homomorphisms from  $p$ -groups into  $G$ . This classification greatly simplifies if  $G$  has a normal Sylow  $p$ -subgroup; the stable homotopy types then depends only on the Weyl group of the Sylow  $p$ -subgroup. If  $G$  is cyclic mod  $p$  then  $BG$  determines  $G$  up to isomorphism. The last class of groups is important because in an appropriate Grothendieck group  $BG$  can be written as a unique linear combination of  $BH$ 's, where  $H$  is cyclic mod  $p$ .

### 0. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Let  $G$  be a finite group. In this note we give a classification of the stable homotopy type of  $BG$  in terms of  $G$ . Our analysis shows that for each prime number  $p$ , the  $p$ -local stable type of  $BG$  depends on the homomorphisms from  $p$ -groups  $Q$  into  $G$ .

The suspension spectrum of  $BG$  and, in particular, its wedge summands have played an important role in homotopy theory. In a previous paper [MP], the authors have given a characterization of the indecomposable summands of  $BG$  in terms of the modular representation theory of  $\text{Out}(Q)$  modules for  $Q < P$  the Sylow subgroup of  $G$ . It is this characterization that we use to study the stable type of  $BG$ . For another such characterization see [BF].

It is known that the stable type of  $BG$  does not determine  $G$  up to isomorphism. A simple example (due to N. Minami) is given by  $Q_{4p} \times \mathbf{Z}/2$  and  $D_{2p} \times \mathbf{Z}/4$  where  $p$  is an odd prime,  $Q_{4p}$  is the generalized quaternion group [CE] of order  $4p$ , and  $D_{2p}$  is the dihedral group of order  $2p$ . The situation is even worse for  $p$ -local classifying spaces since  $BG$  and  $BG/O_{p'}(G)$  have isomorphic mod  $p$  homology and hence equivalent stable types. Here  $O_{p'}(G)$  is the maximal normal subgroup of  $G$  of order prime to  $p$ . However there is a positive result in this direction, due to Nishida [N], who established the following: Suppose  $G_1, G_2$  are finite groups with Sylow  $p$ -subgroups  $P_1, P_2$ , then  $BG_1 \simeq BG_2$  stably at  $p$  implies  $P_1 \approx P_2$ . Our main result is a necessary and sufficient condition.

**Theorem 0.1 (Classification).** *For two finite groups  $G_1, G_2$  the following are equivalent:*

- (1) *Localized at  $p$ ,  $BG_1$  and  $BG_2$  are stably homotopy equivalent.*
- (2) *For every  $p$ -group  $Q$ ,*

$$\mathbf{F}_p\text{Rep}(Q, G_1) \approx \mathbf{F}_p\text{Rep}(Q, G_2)$$

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as  $\text{Out}(Q)$  modules.  $\text{Rep}(Q, G) = \text{Hom}(Q, G)/G$  with  $G$  acting by conjugation.

(3) For every  $p$ -group  $Q$ ,

$$\mathbf{F}_p \text{Inj}(Q, G_1) \approx \mathbf{F}_p \text{Inj}(Q, G_2)$$

as  $\text{Out}(Q)$  modules.  $\text{Inj}(Q, G) < \text{Rep}(Q, G)$  consists of conjugacy classes of injective homomorphisms.

Nishida’s theorem follows since the largest  $Q$  for which  $\text{Inj}(Q, G)$  is nonzero is the Sylow  $p$ -subgroup of  $G$ . We will refer to the common Sylow  $p$ -subgroup as  $P$ . It should not be concluded from (3) that the  $G_1$  conjugacy classes of a  $p$ -subgroup correspond to the  $G_2$  conjugacy classes, although the number of classes is equal.

An important application of Theorem 0.1 is to the case of a normal Sylow  $p$ -subgroup, in particular, the cyclic mod  $p$  groups.

**Definition 0.2.** Two subgroups  $H, K < G$  are called *pointwise conjugate* in  $G$  if there is a bijection of sets  $H \xrightarrow{\alpha} K$  such that  $\alpha(h) = g_h^{-1} h g_h$  for  $g_h \in G$  depending on  $h \in H$ .

Alternately it is easy to see that an equivalent condition is

$$|H \cap (g)| = |K \cap (g)|$$

for all  $g \in G$ , where  $(g)$  denotes the conjugacy class of  $g$ .

Let  $W_G(H)$  denote the *Weyl group* of  $H < G$ , i.e.,

$$W_G(H) = N_G(H)/H \cdot C_G(H)$$

where  $N_G(H)$  is the normalizer and  $C_G(H)$  is the centralizer of  $H$  in  $G$ . Then  $W_G(H) < \text{Out}(H)$ .

**Theorem 0.3.** Suppose  $G_1, G_2$  are finite groups with normal Sylow  $p$ -subgroups  $P_1, P_2$ . Then  $BG_1$  and  $BG_2$  have the same stable homotopy type, localized at  $p$ , if and only if  $P_1 \approx P_2$  ( $\approx P$  say) and  $W_{G_1}(P)$  is pointwise conjugate to  $W_{G_2}(P)$  in  $\text{Out}(P)$ .

**Definition 0.4.**  $G$  is called *cyclic mod  $p$*  (or  $p$ -hypoelementary) if a Sylow  $p$ -subgroup  $P$  is normal and  $C = G/P$  is a cyclic  $p'$ -group, i.e., has order prime to  $p$ . We say  $G$  is *reduced* if  $O_{p'}(G) = 1$ .

For a cyclic mod  $p$  group  $G$ , being reduced is equivalent to  $W_G(P) = C$ .

**Theorem 0.5.** Suppose  $G_1, G_2$  are reduced cyclic mod  $p$  groups. Then  $BG_1 \simeq BG_2$  stably at  $p$  if and only if  $G_1 \approx G_2$ .

Cyclic mod  $p$  groups are important for several reasons. Their mod  $p$  cohomology is computed as the ring of invariants  $H^*(G) = H^*(P)^C$ . On the level of stable homotopy one also has the Minami-Webb Formula [M]: Let  $\mathcal{E}_p(G)$  be the set of cyclic mod  $p$  subgroups of  $G$ . Then

$$BG \simeq \bigvee_{(H)} f(H)/[N_G(H) : H] BH$$

where  $H$  runs over the conjugacy classes of  $\mathcal{E}_p(G)$  and  $f : \mathcal{E}_p(G) \rightarrow \mathbf{Z}$  is the Möbius function given by  $\sum_{J < K} f(K) = 1$  for  $J, K \in \mathcal{E}_p(G)$ .

This expression is to be interpreted in a Grothendieck group of spectra, that is, multiplying through by a common denominator and moving the negative terms to the left side results in a valid formula in the category of  $p$ -local spectra.

The next result shows the uniqueness of the Minami-Webb Formula.

**Theorem 0.6** (Linear independence of cyclic mod  $p$  groups). *Suppose*

$$\bigvee_{i=1}^m \alpha_i BH_i \simeq \bigvee_{j=1}^n \alpha'_j BH'_j$$

for  $H_i, H'_j$  reduced cyclic mod  $p$  groups with  $\alpha_i, \alpha'_j \in \mathbf{Q}$ . Then  $m = n$  and up to a permutation of indices  $\alpha_i = \alpha'_j, H_i = H'_j$ .

The note is organized as follows. In §1 we discuss the classification (Theorem 0.1). In §2 we discuss the normal Sylow  $p$ -subgroup case and explain how it relates to the general case. Section 3 is devoted to cyclic mod  $p$  groups.

Preliminary material and background references for this note may be found in [MP]. All spectra are assumed to be localized at  $p$  and all cohomology groups are taken with simple coefficients in  $\mathbf{F}_p$  unless otherwise noted.

### 1. CLASSIFICATION THEOREM

In this section we briefly discuss Theorem 0.1.

(1) implies (2) since  $\mathbf{F}_p \text{Rep}(Q, G)$  is a quotient of  $\{BQ, BG\} \otimes \mathbf{F}_p$  and the quotient is preserved by stable homotopy equivalences.

The equivalence of (2) and (3) follows from induction on the order of the  $p$ -group  $Q$  and the fact that

$$\mathbf{F}_p \text{Rep}(Q, G) = \bigoplus_R \mathbf{F}_p \text{Surj}(Q, R) \otimes_{\mathbf{F}_p \text{Out}(R)} \mathbf{F}_p \text{Inj}(R, G)$$

where  $R$  runs over isomorphism classes of  $p$ -groups and  $\text{Surj}(Q, R) < \text{Rep}(Q, R)$  consists of conjugacy classes of surjective homomorphisms.

We show that (3) implies (1) by showing that  $BG_1$  and  $BG_2$  have the same indecomposable stable summands with the same multiplicities. If  $X$  is such an indecomposable summand we show that for certain  $p$ -groups  $Q$ ,  $X$  is in one-to-one correspondence with a simple module  $M$  of a quotient ring  $R(Q)$  of the outer endomorphism ring of  $Q$ .  $\mathbf{F}_p \text{Out}(Q)$  is a subring of  $R(Q)$ .

To show statement (1) we need another statement equivalent to statement (3). Let  $K(Q, G) < \text{Inj}(Q, G)$  be the classes of all injective homomorphisms  $\alpha$  such that  $C_G(\text{Im } \alpha)/Z(\text{Im } \alpha)$  is a  $p'$ -group. An inductive argument shows that the statement

$$(4) \quad \mathbf{F}_p K(Q, G_1) \approx \mathbf{F}_p K(Q, G_2)$$

as  $\text{Out}(Q)$  modules, for all  $p$ -groups  $Q$ , is equivalent to statement (3).

For  $\alpha \in \text{Inj}(Q, G)$ , let  $W = N_G(\text{Im } \alpha)/\text{Im } \alpha$ , and  $\overline{W} = \sum_{w \in W} \overline{w}$ , where  $\overline{w}$  means view  $w$  as an element in  $\text{Out}(Q)$ .  $\overline{W} \neq 0 \pmod p$  if and only if  $\alpha \in K(Q, G)$ . For conjugacy classes  $Q_j$  of  $p$ -subgroups of  $G_1$  (or  $G_2$ ), where  $X$  is in one-to-one correspondence with a simple module of  $R(Q_j)$  and the inclusion  $Q_j \hookrightarrow G_1$  (or  $G_2$ ) is in  $K(Q_j, G_i)$ ,  $i = 1$  or  $2$ , we find that the multiplicity of  $X$  in  $BG_1$  (or  $BG_2$ ) equals

$$\sum_j \dim_k \overline{W}_j M_j,$$

where  $k = \text{End}(M_j)$ . Finally we show that if  $W < \text{Out}(Q)$  then

$$\overline{W}M = \text{Hom}(1_W^{\text{Out}(Q)}, P_M) / \text{Hom}(1_W^{\text{Out}(Q)}, \text{Ker}\pi(P_M))$$

where  $P_M \xrightarrow{\pi} M$  is the projective cover of  $M$  as an  $\text{Out}(Q)$  module. Thus the multiplicity depends only on the permutation modules generated by the  $\overline{W}_j$ 's.  $\mathbb{F}_p K(Q, G)$  is a direct sum of such permutation modules, so statement (4) implies statement (1).

## 2. GROUPS WITH NORMAL SYLOW $p$ -SUBGROUPS

In this section we discuss Theorem 0.3 and its relation to Theorem 0.1. We also give an example illustrating Theorem 0.3 and show how the stable and unstable types of classifying spaces differ.

In Theorem 0.3 only a condition involving the Sylow  $p$ -subgroup is given whereas in Theorem 0.1 all  $Q < P$  must be considered. This can be explained as follows. Suppose  $G$  has a normal Sylow  $p$ -subgroup  $P$ . Then  $G = P \rtimes H$  for a  $p'$ -group  $H$  and  $BG \simeq B(G/O_{p'}(G)) = B(P \rtimes W_G(P))$ . Thus we may assume  $G = P \rtimes W_G(P)$  from which we have  $\text{Rep}(Q, G) = \text{Rep}(Q, P) / W_G(P)$ ,  $Q$  a  $p$ -group. Thus

$$\mathbb{F}_p \text{Rep}(Q, G) = \mathbb{F}_p(\text{Rep}(Q, P) / W_G(P)) = \mathbb{F}_p \text{Rep}(Q, P) \otimes_{\text{Out}(P)} 1_{W_G(P)}^{\text{Out}(P)}$$

where  $1_H^G = \mathbb{F}_p(G/H)$ . Character theory implies  $W_{G_1}(P)$  and  $W_{G_2}(P)$  are pointwise conjugate if and only if the induced  $\text{Out}(P)$  modules  $1_{W_{G_1}(P)}^{\text{Out}(P)}$  and  $1_{W_{G_2}(P)}^{\text{Out}(P)}$  are isomorphic. Thus in this case the hypothesis of Theorem 0.3 is equivalent to statement (2) of Theorem 0.1.

**Example 2.1.** Let  $p, l$  be different primes and let  $V$  be an elementary abelian  $p$ -group of rank  $l^n$ . Let  $H_i$  be two nonisomorphic  $l$ -groups of exponent  $l$  and order  $l^n$  (e.g., for  $n = 3, l > 2$ , let  $H_1 = (\mathbb{Z}/l)^n$ ,  $H_2 = U_3(\mathbb{F}_l)$ , the  $3 \times 3$  upper triangular matrices with 1's on the diagonal). These groups act on  $P$  by means of the regular representation

$$H_i \xrightarrow{\text{reg}} \Sigma_{l^n} < \text{GL}_{l^n}(\mathbb{F}_p).$$

It follows that  $G_i = V \rtimes H_i$  are not isomorphic and satisfy  $O_{p'}(G_i) = 1$ . Furthermore  $H_1$  is pointwise conjugate to  $H_2$  in  $\text{GL}(V) = \text{Out}(V)$ . In fact, the nontrivial elements of  $H_1$  and  $H_2$  are all conjugate in  $\Sigma_{l^n}$  since each is an  $l^{n-1}$  fold product of disjoint  $l$  cycles and the conjugacy of elements is determined by their cycle structure. By Theorem 0.3,  $BG_1$  is stably equivalent to  $BG_2$  at  $p$ .

We are indebted to Hans-Werner Henn and Nick Kuhn for pointing out that this example is also interesting because completed at  $p$ ,  $BG_1$  and  $BG_2$  have different (unstable) homotopy types. Thus  $H^*BG_1$  and  $H^*BG_2$  are isomorphic in  $\mathcal{U}$ , the category of unstable modules over the Steenrod algebra  $\mathcal{A}$ , but not in  $\mathcal{A}$ , the category of unstable algebras over  $\mathcal{A}$ .

To prove this we recall the following result of Adams, Miller-Wilkerson, and Lannes [L], [HLS]: For  $G$  a compact Lie group and  $V$  an elementary abelian  $p$ -group, the correspondence  $f \mapsto (Bf)^*$  induces a bijection of  $\text{End}(V)$  sets

$$\text{Rep}(V, G) \approx \text{Hom}_{\mathcal{A}}(H^*BG, H^*BV).$$

Thus an isomorphism of unstable algebras  $H^*BG_i$  would imply an isomorphism of  $\text{End}(V)$  sets  $\text{Rep}(V, G_i) = \text{End}(V)/H_i$  and thus an isomorphism of  $\text{GL}(V)$  sets  $\text{GL}(V)/H_i$ . However, this is false since  $H_1 \not\cong H_2$ .

### 3. CYCLIC mod $p$ GROUPS

In this section we discuss cyclic mod  $p$  groups, which are determined by their classifying spaces, and that their classifying spaces determine all other classifying spaces.

Theorem 0.5 is a direct corollary of Theorem 0.3, since the generators of the Weyl groups will be conjugate.

The uniqueness of the expression in the Minami-Webb formula follows from a result in Harris and Kuhn [HK].

**Definition 3.1.** Let  $G$  be a finite group and  $p$  a prime.

(1) Let  $A_p(G)$  be the Burnside ring generated by isomorphism classes of finite  $G$ -sets  $X$  such that  $X^H = \emptyset$  if  $H$  is not a  $p'$ -subgroup of  $G$ , with addition given by disjoint union.

(2) Let  $R_p(G)$  be the modular representation ring with generators  $[M]$ , where  $M$  is an  $\mathbb{F}_p[G]$ -module, and relations  $[M] = [M_1] + [M_2]$  if there is a short exact sequence  $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$ .

(3) Let  $\psi_G : A_p(G) \rightarrow R_p(G)$  be the map defined by  $\psi_G(X) = \mathbb{F}_p[X]$ .

*N.B.*  $A_p(G)$  is generated by  $G/H$  where  $H$  is a  $p'$ -subgroup and  $\psi_G(G/H) = 1_H^G$ .

**Proposition 3.2 [HK].** For any finite group  $G$ , rank  $\text{Im } \psi_G$  equals the number of conjugacy classes of cyclic  $p'$ -subgroups of  $G$ .

Theorem 0.6 is shown by grading the Grothendieck group of classifying spaces by Sylow  $p$ -subgroups, then using the linear independence of  $p'$ -cyclic permutation modules and their correspondence with classifying spaces of reduced cyclic mod  $p$  groups.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22903

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60208