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FACTORIZATIONS OF INVERTIBLE OPERATORS AND K-THEORY OF C^*-ALGEBRAS

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ABSTRACT. Let $\mathscr{A}$ be a unital C*-algebra. We describe *K-skeleton factorizations* of all invertible operators on a Hilbert C*-module $\mathscr{H}_{\mathscr{A}}$, in particular on $\mathscr{H} = l^2$, with the Fredholm index as an invariant. We then outline the isomorphisms $K_0(\mathscr{A}) \cong \pi_{2k}([p]_0) \cong \pi_{2k}(GL_r^p(\mathscr{A}))$ and $K_1(\mathscr{A}) \cong \pi_{2k+1}([p]_0) \cong \pi_{2k+1}(GL_r^p(\mathscr{A}))$ for $k \geq 0$, where $[p]_0$ denotes the class of all compact perturbations of a projection p in the infinite Grassmann space $Gr^\infty(\mathscr{A})$ and $GL_r^p(\mathscr{A})$ stands for the group of all those invertible operators on $\mathscr{H}_{\mathscr{A}}$ essentially commuting with p.

1. INTRODUCTION

Throughout, we assume that $\mathscr{A}$ is any unital C*-algebra. Let $\mathscr{H}_{\mathscr{A}}$ be the Hilbert (right) $\mathscr{A}$-module consisting of all l^2-sequences in $\mathscr{A}$; i.e., $\mathscr{H}_{\mathscr{A}} := \{\{a_i\} : \sum_{i=1}^\infty a_i^* a_i \in \mathscr{A}\}$, on which an $\mathscr{A}$-valued inner product and a norm are naturally defined by $< \{a_i\}, \{b_i\} >= \sum_{i=1}^\infty a_i^* b_i$ and $\|\{a_i\}\| = \|(\sum_{i=1}^\infty a_i^* a_i)^{1/2}\|$. Let $\mathscr{L}(\mathscr{H}_{\mathscr{A}})$ stand for the C*-algebra consisting of all bounded operators on $\mathscr{H}_{\mathscr{A}}$ whose adjoints exist, and let $\mathscr{K}(\mathscr{H}_{\mathscr{A}})$ denote the closed linear span of all finite rank operators on $\mathscr{H}_{\mathscr{A}}$, respectively. In case $\mathscr{A}$ is the algebra $\mathbf{C}$ of all complex numbers, $\mathscr{H}_{\mathscr{A}}$ is the separable, infinite-dimensional Hilbert space $\mathscr{H} = l^2$; correspondingly, $\mathscr{L}(\mathscr{H}_{\mathscr{A}})$ reduces to the algebra $\mathscr{L}(\mathscr{H})$ of all bounded operators on $\mathscr{H}$, and $\mathscr{K}(\mathscr{H}_{\mathscr{A}})$ reduces to the algebra $\mathscr{K}$ of all compact operators on $\mathscr{H}$. Each element in $\mathscr{L}(\mathscr{H}_{\mathscr{A}})$ can be identified with an infinite, *bounded* matrix whose entries are elements in $\mathscr{A}$ [Zh4, §1]. This identification can be realized by C*-algebraic techniques and the two important *-isomorphisms $\mathscr{L}(\mathscr{H}_{\mathscr{A}}) \cong M(\mathscr{A} \otimes \mathscr{K})$ and $\mathscr{K}(\mathscr{H}_{\mathscr{A}}) \cong \mathscr{A} \otimes \mathscr{K} = (\varinjlim M_n(\mathscr{A}))^-$; where $M(\mathscr{A} \otimes \mathscr{K})$ is the multiplier algebra of $\mathscr{A} \otimes \mathscr{K}$ [Kas]. For more information about multiplier algebras the reader is referred to [APT, Bl, Cu1, El, Br2, Pe1,

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OP, L, Zh4–5], among others. The set of projections

$$Gr^{\infty}(\mathscr{A}) := \{p \in \mathscr{L}(\mathscr{H}_{\mathscr{A}}) : \ p = p^2 = p^* \ \text{ and } \ p \sim 1 \sim 1 - p\}$$

is called the *infinite Grassmann space associated with* $\mathscr{A}$ *;* where '$q \sim p$' is the well-known Murray-von Neumann equivalence of two projections; i.e., there exists a partial isometry $v \in \mathscr{L}(\mathscr{H}_{\mathscr{A}})$ such that $vv^* = p$ and $v^*v = q$. If $\mathscr{A} = \mathbf{C}$, then $Gr^{\infty}(\mathscr{A})$ reduces to the well-known Grassmann space $Gr^{\infty}(\mathscr{H})$ consisting of all projections on $\mathscr{H}$ with an infinite dimension and an infinite codimension.

2. Factorizations and K-theory

Let $p \in Gr^{\infty}(\mathscr{A})$. If x is any element in $\mathscr{L}(\mathscr{H}_{\mathscr{A}})$, with respect to the decomposition $p \oplus (1-p) = 1$ one can write x as a 2×2 matrix, say $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $a = pxp$, $b = px(1-p)$, $c = (1-p)xp$, and $d = (1-p)x(1-p)$. A unitary operator $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is called a *K-skeleton unitary along* p, if both b and c are some partial isometries in $\mathscr{A} \otimes \mathscr{K}$. An easy calculation shows that a unitary operator u is a K-skeleton unitary if and only if a is a Fredholm partial isometry on the submodule $p\mathscr{H}_{\mathscr{A}}$ and d is a Fredholm partial isometry on the submodule $(1-p)\mathscr{H}_{\mathscr{A}}$; in other words, all $p - aa^*$, $p - a^*a$, $(1-p) - dd^*$, $(1-p) - d^*d$ are projections in $\mathscr{A} \otimes \mathscr{K}$. The term '$K$-skeleton' is chosen, since $K_0(\mathscr{A})$ is completely described by the homotopy classes of all such unitaries.

Let $GL_r^p(\mathscr{A})$ be the topological group consisting of all those invertible operators in $\mathscr{L}(\mathscr{H}_{\mathscr{A}})$ such that $xp - px \in \mathscr{A} \otimes \mathscr{K}$, equipped with the norm topology from $\mathscr{L}(\mathscr{H}_{\mathscr{A}})$. Let $GL_{\infty}^p(\mathscr{A})$ stand for the path component of $GL_r^p(\mathscr{A})$ containing the identity; in the special case when $\mathscr{A} = \mathbf{C}$, we instead use the notation $GL_r^p(\mathscr{H})$ and $GL_{\infty}^p(\mathscr{H})$, respectively. Let $GL_{\infty}(\mathscr{A})$ and $GL_{\infty}^0(\mathscr{A})$ denote the group of all invertible elements in the unitization of $\mathscr{A} \otimes \mathscr{K}$ and its identity path component, respectively.

2.1. **K-skeleton factorization theorem** [Zh4]. (i) *If* $x \in GL_r^p(\mathscr{A})$, *then there exist an element* $k \in \mathscr{A} \otimes \mathscr{K}$, *an invertible element* $\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}\right)$, *and a K-skeleton unitary* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *along* p *such that* $1 + k \in GL_{\infty}^0(\mathscr{A})$ *and*

$$x = (1+k)\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A factorization of x *with the form above is called a* K*-skeleton factorization along* p.

(ii) If two K-skeleton factorizations of x along p are given, say

$$x = x_0 x_p \begin{pmatrix} a & b \\ c & d \end{pmatrix} = x_o' x_p' \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

then $[cc^*] - [bb^*] = [c'c'^*] - [b'b'^*] \in K_0(\mathscr{A})$; in other words, $[cc^*] - [bb^*]$ is an invariant independent of all (infinitely many) possible K-skeleton factorizations of x along p.

Outline of a proof. There is a shorter proof solely for this theorem. For the sake of clarifying some internal relations among $\pi_0(GL_r^p(\mathscr{A}))$, $\pi_0([p]_0)$, and $K_0(\mathscr{A})$, we outline a proof as follows. First, every element in $GL_{\infty}^p(\mathscr{A})$ can be written as a product of the form $x_0 x_p$ for some invertible $x_0 \in GL_{\infty}^0(\mathscr{A})$ with

$x_0 - 1 \in \mathscr{A} \otimes \mathscr{K}$ and another invertible x_p with $x_p p = p x_p$ [Zh4]. Secondly, write the polar decomposition $x = (xx^*)^{1/2} u$, where $(xx^*)^{1/2} \in GL^p_\infty(\mathscr{A})$ and u is a unitary in $GL^p_r(\mathscr{A})$. Then consider the following subsets of $Gr^\infty(\mathscr{A})$:

$$[upu^*]_r := \{wupu^*w^* : w \in GL^0_\infty(\mathscr{A}) \text{ with } ww^* = w^*w = 1\}$$

and

$$[p]_0 := \{vpv^* : v \in GL^p_r(\mathscr{A}) \quad vv^* = v^*v = 1\}.$$

Technical arguments show that $[upu^*]_r$ is precisely the path component of $[p]_0$ containing upu^*. Thirdly, there is a representative in $[upu^*]_r$ with the form $(p - r_1) \oplus r_2$ for some projections $r_1, r_2 \in \mathscr{A} \otimes \mathscr{K}$. It follows that there exists a unitary $u_0 \in GL^0_\infty(\mathscr{A})$ such that

$$u_0^* upu^* u_0 = (p - r_1) \oplus r_2.$$

Then one obtains a K-skeleton unitary $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $u = u_0 \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $bb^* = r_1$ and $cc^* = r_2$. Since $(xx^*)^{1/2} u_0 \in GL^p_\infty(\mathscr{A})$, we can rewrite it as a product in the desired form $x_0 \left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}\right)$. The details are contained in [Zh4].

It follows from Theorem 2.1 that $x \cdot GL^p_\infty(\mathscr{A}) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot GL^p_\infty(\mathscr{A})$ (cosets) for each $x \in GL^p_r(\mathscr{A})$. The invariant $[cc^*] - [bb^*]$ associated with the K-skeleton factorization of $x \in GL^p_r(\mathscr{A})$ yields the bijection

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot GL^p_\infty(\mathscr{A}) \longleftrightarrow [(p - bb^*) \oplus (cc^*)]_r.$$

It can be shown that $[(p - r_1) \oplus r_1']_r = [(p - r_2) \oplus r_2']_r$ iff $[r_1'] - [r_1] = [r_2'] - [r_2]$ in $K_0(\mathscr{A})$. Therefore, we conclude the following theorem whose details are given in [Zh4].

2.2. **Theorem** [Zh4]. *The maps defined by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot GL^p_\infty(\mathscr{A}) \longmapsto [(p - r_1) \oplus r_2]_r \longmapsto [r_2] - [r_1]$$

are two bijections, which induce the following isomorphisms:

$$GL^p_r(\mathscr{A}) / GL^p_\infty(\mathscr{A}) \cong D_h([p]_0) \cong K_0(\mathscr{A}),$$

where $GL^p_r(\mathscr{A}) / GL^p_\infty(\mathscr{A})$ is the quotient group with the induced multiplication and

$$D_h([p]_0) = \{[upu^*]_r : u \in GL^p_r(\mathscr{A}) \text{ with } uu^* = u^*u = 1\}$$

is the set of all path components of $[p]_0$. The group operation on $D_h([p]_0)$ is defined by

$$[(p - r_1) \oplus r_1']_r + [(p - r_2) \oplus r_2']_r = [(p - r_1 - s_2) \oplus (r_1' \oplus s_2')]_r$$

for some projections $s_2 \in p(\mathscr{A} \otimes \mathscr{K})p$ and $s_2' \in (1 - p)(\mathscr{A} \otimes \mathscr{K})(1 - p)$ such that $s_2 \sim r_2$, $s_2 r_1 = 0$, $s_2' \sim r_2'$, and $s_2' r_1' = 0$.

2.3. **Theorem.** *Let the base point of $[p]_0$ be p and the base point of $GL^p_r(\mathscr{A})$ be the identity. Then*

$$\pi_{2k+1}([p]_0) \cong \pi_{2k+1}(GL^p_r(\mathscr{A})) \cong K_1(\mathscr{A}),$$

and

$$\pi_{2k+2}([p]_0) \cong \pi_{2k+2}(GL_r^p(\mathscr{A})) \cong K_0(\mathscr{A}) \quad \forall\, k \geq 0.$$

Outline of a proof. Let $U_\infty(\mathscr{A})$ be the unitary group of the unitization of $\mathscr{A} \otimes \mathscr{K}$, and let $U_p(\mathscr{A})$ be the subgroup of $U_\infty(\mathscr{A})$ consisting of all those unitaries commuting with p. First, the map $\psi_p : U_\infty(\mathscr{A}) \longrightarrow [p]_r$ defined by $\psi_p(u) = upu^*$ is a Serre (weak) fibration with a standard fiber $U_p(\mathscr{A})$ [Zh6, §2]. Secondly, the long exact sequence of homotopy groups associated with this fibration breaks into short exact sequences [Zh6, 2.5, 2.8]:

$$0 \longrightarrow \pi_{k+1}([p]_r) \longrightarrow \pi_k(U_p(\mathscr{A})) \longrightarrow \pi_k(U_\infty(\mathscr{A})) \longrightarrow 0 \qquad (k \geq 0).$$

Thirdly, by an analysis on this short exact sequence one concludes

$$\pi_{2k+2}([p]_0) \cong K_0(\mathscr{A}) \quad \text{and} \quad \pi_{2k+1}([p]_0) \cong K_1(\mathscr{A}) \qquad (k \geq 0).$$

It is well known that the subgroup $U_r^p(\mathscr{A})$ consisting of all unitary elements in $GL_r^p(\mathscr{A})$ is homotopy equivalent to $GL_r^p(\mathscr{A})$. We consider the maps $U_r^p(\mathscr{A}) \longrightarrow [p]_0$ defined by $\phi_p(u) = upu^*$. It can be shown that ϕ_p is a weak fibration with a standard fiber $U^p(\mathscr{A})$, where $U^p(\mathscr{A})$ is the group consisting of all those unitaries in $U_r^p(\mathscr{A})$ commuting with p. An argument similar to that above applies to this fibration. One can show that $\pi_{2k+1}(U_r^p(\mathscr{A})) \cong K_1(\mathscr{A})$ and $\pi_{2k+2}(U_r^p(\mathscr{A})) \cong K_0(\mathscr{A})$ for $k \geq 0$. The details are given in [Zh6, §4].

2.4. **Special case** $\mathscr{A} = C(X)$. In particular, if $\mathscr{A}$ is taken to be the commutative C*-algebra $C(X)$ consisting of all complex-valued continuous functions on a compact Hausdorff space X, then each element in $\mathscr{L}(\mathscr{H}_{C(X)})$ can be identified with a norm-bounded, *-strong continuous map from X to $\mathscr{L}(\mathscr{H})$ [APT]. Here $\mathscr{L}(\mathscr{H}) \supset \{x_\lambda\}$ converges to x in the *-strong operator topology iff

$$\|(x_\lambda - x)k\| + \|k(x_\lambda - x)\| \to 0 \quad \text{for any } k \in \mathscr{K}.$$

Obviously, $\mathscr{L}(\mathscr{H}_{C(X)})$ contains the C*-tensor product $\mathscr{L}(\mathscr{H})\otimes C(X)$ consisting of all norm-continuous maps from X to $\mathscr{L}(\mathscr{H})$ as a C*-subalgebra. Then Theorems 2.1 and 2.2 in this special case are interpreted as follows.

2.5. **Corollary.** *Let $GL^\infty(\mathscr{H})$ be the group of all invertible operators in $\mathscr{L}(\mathscr{H})$.*

(i) *If $f : X \longrightarrow GL^\infty(\mathscr{H})$ is a norm-bounded, *-strong continuous map and p is a projection in the infinite Grassmann space $Gr^\infty(\mathscr{H})$ such that $pf - fp \in \mathscr{K} \otimes C(X)$, then f can be factored as the following product of three invertible maps*

$$f(.) = \begin{pmatrix} 1 + k_{11}(.) & k_{12}(.) \\ k_{21}(.) & 1 + k_{22}(.) \end{pmatrix} \begin{pmatrix} g_1(.) & 0 \\ 0 & g_2(.) \end{pmatrix} \begin{pmatrix} a(.) & b(.) \\ c(.) & d(.) \end{pmatrix};$$

*where $k_{ij}(.)$'s are norm-continuous maps from X to $\mathscr{K}$, $g_1(.) \oplus g_2(.)$ is a norm-bounded, *-strong continuous map from X to $GL^\infty(\mathscr{H})$, $a(.)$, $d(.)$ are *-strong continuous maps from X to the set of Fredholm partial isometries on $p\mathscr{H}$ and $(1-p)\mathscr{H}$, respectively, and $c(.)$, $b(.)$ are norm-continuous maps from X to the set of partial isometries in $\mathscr{H}$. Furthermore,*

$$[c(.)c(.)^*] - [b(.)b(.)^*] \in K_0(C(X)) \; (\cong K^0(X))$$

is an invariant independent of all possible factorization with the above form.

(ii) *The groups* $[X, GL_r^p(\mathscr{H})]$, $[X, [p]_0]$, *and* $K_0(C(X))$ *are isomorphic, where* $[X, .]$ *is the set of homotopy classes of norm-bounded, *-strong continuous maps from* X *to* $(.)$.

2.6. **Invertible dilations of a Fredholm operator.** Let us illustrate a K-skeleton factorization of any invertible dilation of a Fredholm operator $x \in \mathscr{L}(\mathscr{H}_{\mathscr{A}})$. There are of course infinitely many invertible 2×2 matrices with the form

$$D_2(x) := \begin{pmatrix} x & y_1 \\ y_2 & z \end{pmatrix} \in M_2(\mathscr{L}(\mathscr{H}_{\mathscr{A}})).$$

Each such 2×2 invertible matrix is called *an invertible dilation of* x. Specific constructions of such a dilation were given by P. Halmos [Ho, 222] and A. Connes [Co]. For each invertible dilation of x it follows from the K-skeleton Factorization Theorem 2.1 that

$$\begin{pmatrix} x & y_1 \\ y_2 & z \end{pmatrix} = \begin{pmatrix} 1 + a_{11} & a_{12} \\ a_{21} & 1 + a_{22} \end{pmatrix} \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \begin{pmatrix} v & 1 - vv^* \\ 1 - v^*v & -v^* \end{pmatrix},$$

where a_{ij}'s are some elements in $\mathscr{A} \otimes \mathscr{K}$, $z_1, z_2 \in GL^\infty(\mathscr{A})$, and the above matrix on the right, say w, is a familiar unitary matrix occurring in the index map in K-theory [Bl, 8.3.2] in which v is a Fredholm partial isometry in $\mathscr{L}(\mathscr{H}_{\mathscr{A}})$. Set $p = \operatorname{diag}(1, 0)$. It is well known that

$$[1 - v^*v] - [1 - vv^*] \in K_0(\mathscr{A})$$

is precisely the Fredholm index $\operatorname{Ind}(v) = \operatorname{Ind}(pxp)$ (on $p\mathscr{H}_A$). It follows from Theorem 2.1(ii) that those K-skeleton unitaries associated with all possible invertible dilations of x in $M_2(\mathscr{L}(\mathscr{H}_{\mathscr{A}}))$ only differ from w by a factor in $GL_\infty^p(\mathscr{A})$.

3. Factorizations of invertible operators with integer indices

Now we consider some special cases such that $K_0(\mathscr{A}) \cong Z$ (the group of all integers); for example, $\mathscr{A} = \mathbf{C}$, or $\mathscr{A} = C(S^{2n+1})$ where S^m is the standard m-sphere, or $\mathscr{A} = \mathscr{O}_\infty$, the Cuntz algebra generated by isometries $\{s_i\}_{i=1}^\infty \subset \mathscr{L}(\mathscr{H})$ such that $\sum_{i=1}^\infty s_i s_i^* \le 1$.

Let p be any projection in $Gr^\infty(\mathscr{H}) \subset Gr^\infty(\mathscr{A})$ [the inclusion holds because $\mathscr{H} \subset \mathscr{H}_{\mathscr{A}}$ and $\mathscr{L}(\mathscr{H}) \subset \mathscr{L}(\mathscr{H}_{\mathscr{A}})$]. Let $\{\xi_i\}_{i=0}^{+\infty}$ be any orthonormal basis of the subspace $p\mathscr{H}$ and $\{\xi_i\}_{-\infty}^{i=-1}$ be any orthonormal basis of the subspace $(1-p)\mathscr{H}$. Then $\{\xi_i\}_{-\infty}^{+\infty}$ is an orthonormal basis of both $\mathscr{H}$ and $\mathscr{H}_{\mathscr{A}}$. Let u_0 denote the bilateral shift associated with the basis $\{\xi_i\}_{-\infty}^{+\infty}$ of $\mathscr{H}$, defined by $u_0(\xi_i) = \xi_{i+1}$ for all $i \in Z$. Clearly, u_0 is a K-skeleton unitary of $\mathscr{L}(\mathscr{H}_{\mathscr{A}})$ along p. Applying the K-skeleton Factorization Theorem 2.1 to the above special cases, we have the following factorizations of invertible operators orientated by the integer-valued Fredholm index:

3.1. **Corollary.** *Suppose that* $K_0(\mathscr{A}) \cong Z$ *is generated by* $[1]$ *where* 1 *is the identity of* $\mathscr{A}$. *If* x *is an invertible operator on* $\mathscr{H}_{\mathscr{A}}$ *such that* $px - xp \in \mathscr{A} \otimes \mathscr{K}$, *then* $x = (1 + k)x_p u_0^{-n}$, *where* $k \in \mathscr{A} \otimes \mathscr{K}$, x_p *is an invertible operator commuting with* p, *and the integer* n *is the Fredholm index of* pxp *on the submodule* $p\mathscr{H}_{\mathscr{A}}$, *say* $\operatorname{Ind}(pxp)$, *which is independent of the choice of* $\{\xi_i\}_{i=0}^{+\infty}$, $\{\xi_i\}_{-\infty}^{-1}$ *and all possible factorizations along* p *with the same form above.*

Outline of a proof. It is obvious that $\operatorname{Ind}(pu_0^n p) = -n$. Let G be the group $\{u_0^n : n \in Z\}$ in which every element is a K-skeleton unitary along p. As a special case of Theorem 2.1 one can show that the map from G to $GL_r^p(\mathscr{A})/GL_\infty^p(\mathscr{A})$ defined by $u_0^n \longmapsto u_0^n \cdot GL_\infty^p(\mathscr{A})$ is a group isomorphism. It follows that $\pi_0(GL_r^p(\mathscr{A})) = \{u_0^n \cdot GL_\infty^p(\mathscr{A}) : n \in Z\}$. Then the factorization follows. The reader may want to consider the extreme case $\mathscr{A} = \mathbf{C}$ and then generalize the conclusion to a larger class of C*-algebras.

A similar proof yields the following alternative factorization of x as a product of three invertibles under the same assumptions as of Corollary 3.1:

$$x = \begin{cases} (1+k_1)x_1 & \text{if } \operatorname{Ind}(pxp) = 0, \\ (1+k_2)x_2(u_1 \oplus u_2 \oplus \cdots \oplus u_{-n} \oplus w_1) & \text{if } \operatorname{Ind}(pxp) = n < 0, \\ (1+k_3)x_3(u_1^* \oplus u_2^* \oplus \cdots \oplus u_n^* \oplus w_2) & \text{if } \operatorname{Ind}(pxp) = n > 0, \end{cases}$$

where u_i is a bilateral shift on a subspace $\mathscr{H}_i$ of $\mathscr{H}$ for $1 \le i \le n$, w_j's are unitary operators on $(\bigoplus_{i=1}^n \mathscr{H}_i)^\perp$, $k_j \in \mathscr{A} \otimes \mathscr{K}$, and x_j's are invertible operators commuting with p.

3.2. **Corollary.** *Suppose that* $K_0(\mathscr{A}) \cong Z$ *is generated by* $[1]$. *If* x *is an arbitrary* element $\mathscr{L}(\mathscr{H}_\mathscr{A})$ and $p \in Gr^\infty(\mathscr{H})$ (as above) such that $px - xp \in \mathscr{A} \otimes \mathscr{K}$, then there exists a *unique* norm-continuous map $x(\lambda)$ from $C \setminus \sigma(x)$ to $GL_\infty^p(\mathscr{A})$, where $\sigma(x)$ is the spectrum of x, such that $x - \lambda = x(\lambda)u_0^{-n_i}$, where $n_i = \operatorname{Ind}(p(x-\lambda_i)p)$ and λ_i is any complex number in the ith path component O_i of $C \setminus \sigma(x)$. An alternative K-skeleton factorization of $x - \lambda$ for $\lambda \in O_i$ is as follows (when $n_i \neq 0$):

$$x - \lambda = \begin{cases} y_i(\lambda)(u_1 \oplus u_2 \oplus \cdots \oplus u_{|n_i|} \oplus w_i) & \text{if } \operatorname{Ind}(p(x-\lambda_i)p) = n_i < 0, \\ y_i'(\lambda)(u_1^* \oplus u_2^* \oplus \cdots \oplus u_{n_i}^* \oplus v_i) & \text{if } \operatorname{Ind}(p(x-\lambda_i)p) = n_i > 0, \end{cases}$$

where u_i's are bilateral shifts on mutually orthogonal closed subspaces $\mathscr{H}_i$'s of $\mathscr{H}$, w_i, v_i's are unitary operators on the subspace $(\bigoplus_{i=1}^{|n_i|} \mathscr{H}_i)^\perp$, and $y_i(\lambda)$, $y_i'(\lambda)$ are norm-continuous maps from O_i to $GL_\infty^p(\mathscr{A})$.

3.3. **Winding numbers of invertible operators.** Using the first factorization in Corollary 3.2, we assign an integer n_i to each path component O_i of $C \setminus \sigma(x)$, which is precisely the minus winding number of $u_0^{-n_i}$ as a continuous map from S^1 to S^1 (via the Gel'fand transformation). We call n_i the *winding number of* x *along* p *over* O_i. As a particular case, if x is an operator whose essential spectrum, the spectrum of $\pi(x)$ in the generalized Calkin algebra $\mathscr{L}(\mathscr{H}_\mathscr{A})/\mathscr{K}(\mathscr{H}_\mathscr{A})$, does not separate the plane, then all winding numbers of x along any $p \in Gr^\infty(\mathscr{A})$ are zero as long as $px - xp \in \mathscr{A} \otimes \mathscr{K}$. There is another way to describe the integer n_i.

3.4. **Corollary.** *Let* $G_i(x)$ *denote the subgroup of* $GL_r^p(\mathscr{A})$ *generated by* $GL_\infty^p(\mathscr{A})$ *and* $x - \lambda_i$ *where* $\lambda_i \in O_i$. *Then* $G_i(x)/GL_\infty^p(\mathscr{A}) \cong n_i Z$, *and hence* $GL_r^p(\mathscr{A})/G_i(x) \cong Z_{n_i}$, *the finite cyclic group of order* n_i.

In particular, one can apply the above factorizations to an invertible dilation of a pseudodifferential operator of order zero on a compact manifold and classical multiplication operators. Let us spend few lines to look at the following familiar examples.

3.5. **Multiplication operators.** Let M_f be the invertible multiplication operator with symbol f in $L^\infty(S^1)$, where S^1 is the unit circle; i.e., $M_f(g) = fg$ for any $g \in L^2(S^1)$. If p is a projection on $L^2(S^1)$ such that $\dim(p) = \operatorname{codim}(1-p) = \infty$ and $pM_f - M_fp$ is a compact operator, then it follows from Corollary 3.1 that $M_f = (1+k)x_pu_0^{-n}$, where $n = \operatorname{Ind}(pM_fp)$, k is a compact operator on $L^2(S^1)$, x_p is an invertible operator on $L^2(S^1)$ commuting with p, and u_0 is a bilateral shift operator associated with a fixed orthonormal basis of $L^2(S^1)$. It is well known that pM_fp is a familiar Toeplitz operator on the subspace $pL^2(S^1)$.

3.6. **Restricted loop group along** $p \in Gr^\infty(\mathscr{H})$. Consider the following *restricted loop group along* p consisting of all norm-bounded, *-strong continuous maps from S^1 to $GL_r^p(\mathscr{H})$, denoted by $\operatorname{Map}(S^1, GL_r^p(\mathscr{H}))_\beta$. Since $K_0(C(S^1)) = Z$, each $f \in \operatorname{Map}(S^1, GL_r^p(\mathscr{H}))_\beta$ can be factored as $f = (1+f_0)f_1u_0^{-n}$, where $n = \operatorname{Ind}(pfp)$, f_0 is a norm-continuous map from S^1 to $\mathscr{K}$, f_1 is a *-strong continuous map from S^1 to $GL^\infty(\mathscr{H})$ such that $f_1(z)p = pf_1(z)$ for any $z \in S^1$, and u_0 is a bilateral shift with respect to a fixed orthonormal basis of $\mathscr{H}$. If f is norm-continuous, then f_1 is also norm continuous. Furthermore, $[S^1, GL_r^p(\mathscr{H})] \cong [X, [p]_0] \cong Z$. The same conclusions also hold, if S^1 is replaced by S^{2n+1} for any $n \geq 1$.

3.7. *Remarks.* (i) Theorems 2.1–2.3 still hold, if $\mathscr{A}$ is any stably unital C*-algebra; i.e., $\mathscr{A} \otimes \mathscr{K}$ has an approximate identity consisting of a sequence of projections [Bl, 5.5.4; Zh4].

(ii) Let $\operatorname{Index}(x, p)$ denote the invariant $[cc^*] - [bb^*] \in K_0(\mathscr{A})$ in Theorem 2.1(ii). If p is fixed, then $\operatorname{Index}(x, p)$ is precisely the Fredholm index of pxp as an operator on $p\mathscr{H}_{\mathscr{A}}$ and fits into the established theory of the $K_0(\mathscr{A})$-valued Fredholm index. However, some new results do arise from invariants of $\operatorname{Index}(x, p)$ as the variable p runs in $\{p \in Gr^\infty(\mathscr{A}) : xp - px \in \mathscr{A} \otimes \mathscr{K}\}$ or as x and p jointly change [Zh7]. As a matter of fact, $\operatorname{Index}(x, p)$ is an invariant under homotopy and perturbation by elements in $\mathscr{A} \otimes \mathscr{K}$ with respect to both variables x and p. For example, by the combination of the K-skeleton Factorization Theorem and certain invariants of $\operatorname{Index}(x, p)$, we proved [Zh7] the following:

Theorem.

$$\pi_0(GL(M_n(\mathbf{C})'_e)) \cong \{k \in K_0(\mathscr{A}) : n \cdot k = 0\} \quad \textit{for any } n \geq 2\,;$$

where $GL(M_n(\mathbf{C})'_e)$ *denotes the group of all invertibles in the essential commutant* $M_n(\mathbf{C})'_e$ *of* $M_n(\mathbf{C})$ *which is naturally embedded in* $M_n(\mathscr{L}(\mathscr{H}_{\mathscr{A}}))$.

(iii) The reader may want to compare (3.1)–(3.3) and the famous BDF theory [BDF1,2] to see their obvious relations; we work with invertibles on $\mathscr{H}_{\mathscr{A}}$, while the BDF theory dealt with Fredholm operators.

(iv) In [PS] Pressley and Segal have studied the *restricted general linear group*

$$GL_{\text{res}}(\mathscr{H}) := \{x \in GL^\infty(\mathscr{H}) : xp - px \textit{ is Hilbert-Schmidt}\}$$

and given some applications to the Kdv equations. It is a hope that our results will shed some light in the same direction.

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