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LINKLESS EMBEDDINGS OF GRAPHS IN 3-SPACE

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ABSTRACT. We announce results about flat (linkless) embeddings of graphs in 3-space. A piecewise-linear embedding of a graph in 3-space is called *flat* if every circuit of the graph bounds a disk disjoint from the rest of the graph. We have shown:

(i) An embedding is flat if and only if the fundamental group of the complement in 3-space of the embedding of every subgraph is free.

(ii) If two flat embeddings of the same graph are not ambient isotopic, then they differ on a subdivision of K_5 or $K_{3,3}$.

(iii) Any flat embedding of a graph can be transformed to any other flat embedding of the same graph by "3-switches", an analog of 2-switches from the theory of planar embeddings. In particular, any two flat embeddings of a 4-connected graph are either ambient isotopic, or one is ambient isotopic to a mirror image of the other.

(iv) A graph has a flat embedding if and only if it has no minor isomorphic to one of seven specified graphs. These are the graphs that can be obtained from K_6 by means of $Y\Delta$- and ΔY-exchanges.

1. INTRODUCTION

All spatial embeddings are assumed to be piecewise linear. If C, C' are disjoint simple closed curves in S^3, then their *linking number*, $\text{lk}(C, C')$, is the number of times (mod 2) that C crosses over C' in a regular projection of $C \cup C'$. In this paper graphs are finite, undirected, and may have loops and multiple edges. Every graph is regarded as a topological space in the obvious way. We say that an embedding of a graph G in S^3 is *linkless* if every two disjoint circuits of G have zero linking number. The following is a result of Sachs [13, 14] and Conway and Gordon [3].

(1.1) *The graph K_6 (the complete graph on six vertices) has no linkless embedding.*

Proof. Let ϕ be an embedding of K_6 into S^3. By studying the effect of a crossing change in a regular projection, it is easy to see that the mod 2 sum $\sum \text{lk}(\phi(C_1), \phi(C_2))$, where the sum is taken over all unordered pairs of disjoint circuits C_1, C_2 of K_6, is an invariant independent of the embedding. By checking an arbitrary embedding we can establish that this invariant equals 1. □

Let G be a graph and let v be a vertex of G of valency 3 with distinct neighbors. Let H be obtained from G by deleting v and adding an edge

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between every pair of neighbors of v. We say that H is obtained from G by a *$Y\Delta$-exchange* and that G is obtained from H by a *ΔY-exchange*. The *Petersen family* is the set of all graphs that can be obtained from K_6 by means of $Y\Delta$- and ΔY-exchanges. There are exactly seven such graphs, one of which is the Petersen graph. Pictures of these graphs can be found in [13–15]. Sachs [13, 14] has in fact shown that no member of the Petersen family has a linkless embedding [the argument is similar to the proof of (1.1)] and raised the problem of characterizing linklessly embeddable graphs. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. It is easy to see that the property of having a linkless embedding is preserved under taking minors, and that led Sachs to conjecture that a graph is linklessly embeddable if and only if it has no minor in the Petersen family. We have shown that this is true. Moreover, let us say that an embedding ϕ of a graph G in S^3 is *flat* if for every circuit C of G there exists an open disk in S^3 disjoint from $\phi(G)$ whose boundary is $\phi(C)$. Clearly every flat embedding is linkless, but the converse need not hold. However, Böhme [1] and Saran [15] conjectured that a graph has a linkless embedding if and only if it has a flat one. This is also true, for we have shown the following.

(1.2) *For a graph G, the following are equivalent*:

- (i) *G has a flat embedding,*
- (ii) *G has a linkless embedding,*
- (iii) *G has no minor in the Petersen family.*

There have been a number of other attempts [8, 15, 2] at proving (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii). However, none of them is correct.

For the proof of (1.2) we need the following two theorems, which may be of independent interest.

(1.3) *Let ϕ be an embedding of a graph G in S^3. Then ϕ is flat if and only if for every subgraph G' of G, the fundamental group of $S^3 - \phi(G')$ is free.*

Let ϕ_1, ϕ_2 be two embeddings of a graph G in S^3. We say that ϕ_1, ϕ_2 are *ambient isotopic* if there exists an orientation preserving homeomorphism h of S^3 onto S^3 such that $\phi_1 = h\phi_2$. (We remark that by a result of Fisher [4] h can be realized by an ambient isotopy.) If ϕ is an embedding of a graph G in S^3 we denote by $-\phi$ the embedding of G obtained by composing ϕ with the antipodal map.

(1.4) *Let G be a 4-connected graph and let ϕ_1, ϕ_2 be two flat embeddings of G. Then ϕ_1 is ambient isotopic to either ϕ_2 or $-\phi_2$.*

2. The fundamental group

A basic tool for working with flat embeddings is the following lemma of Böhme [1] (see also [15]).

(2.1) *Let ϕ be a flat embedding of a graph G into S^3, and let $C_1, C_2, \ldots, C_n$ be a family of circuits of G such that for every $i \neq j$, the intersection of C_i and C_j is either connected or null. Then there exist pairwise disjoint open disks $D_1, D_2, \ldots, D_n$, disjoint from $\phi(G)$ and such that $\phi(C_i)$ is the boundary of D_i for $i = 1, 2, \ldots, n$.*

We illustrate the use of (2.1) with the following, which is a special case of a theorem of Wu [18]. An embedding ϕ of a graph G in S^3 is *spherical* if there exists a surface $\Sigma \subseteq S^3$ homeomorphic to S^2 such that $\phi(G) \subseteq \Sigma$. Clearly if ϕ is spherical then G is planar.

(2.2) *Let ϕ be an embedding of a planar graph G in S^3. Then ϕ is flat if and only if it is spherical.*

Proof. Clearly if ϕ is spherical then it is flat. We prove the converse only for the case when G is 3-connected. Let $C_1, C_2, \dots, C_n$ be the collection of face-boundaries in some planar embedding of G. These circuits satisfy the hypothesis of (2.1). Let $D_1, D_2, \dots, D_n$ be the disks as in (2.1); then $\phi(G) \cup D_1 \cup D_2 \cup \cdots \cup D_n$ is the desired sphere. □

The following is a result of Scharlemann and Thompson [16].

(2.3) *Let ϕ be an embedding of a graph G in S^3. Then ϕ is spherical if and only if*

(i) *G is planar, and*
(ii) *for every subgraph G' of G, the fundamental group of $S^3 - \phi(G')$ is free.*

We see that by (2.2), (1.3) is a generalization of (2.3). In fact, we prove (1.3) by reducing it to planar graphs and then applying (2.3). Let us prove the "only if" part of (1.3). Let G' be a subgraph of G such that $\pi_1(S^3 - \phi(G'))$ is not free. Choose a maximal forest F of G' and let G'' be obtained from G' by contracting all edges of F, and let ϕ'' be the induced embedding of G''. Then $\pi_1(S^3 - \phi''(G'')) = \pi_1(S^3 - \phi(G'))$ is not free, but G'' is planar, and so ϕ'' is not flat by (2.2) and (2.3). Hence ϕ is not flat, as desired.

Let G be a graph, and let e be an edge of G. We denote by $G\backslash e(G/e)$ the graph obtained from G by deleting (contracting) e. If ϕ is an embedding of G in S^3, then it induces embeddings of $G\backslash e$ and (up to ambient isotopy) of G/e in the obvious way. We denote these embeddings by $\phi\backslash e$ and ϕ/e, respectively.

(2.4) *Let ϕ be an embedding of a graph G into S^3, and let e be a nonloop edge of G. If both $\phi\backslash e$ and ϕ/e are flat, then ϕ is flat.*

Proof. Suppose that ϕ is not flat. By (1.3) there exists a subgraph G' of G such that $\pi_1(S^3 - \phi(G'))$ is not free. If $e \notin E(G')$ then $\phi\backslash e$ is not flat by (1.3). If $e \in E(G')$ then ϕ/e is not flat by (1.3), because $\pi_1(S^3 - (\phi/e)(G'/e)) = \pi_1(S^3 - \phi(G'))$ is not free. □

We say that a graph G is a *coforest* if every edge of G is a loop. The following follows immediately from (2.4).

(2.5) *Let ϕ be an embedding of a graph G in S^3. Then ϕ is flat if and only if the induced embedding of every coforest minor of G is flat.*

3. Uniqueness

A graph H is a *subdivision* of a graph G if H can be obtained from G by replacing edges by pairwise internally-disjoint paths. We recall that Kuratowski's theorem [6] states that a graph is planar if and only if it contains no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$. It follows from a theorem of Mason [7] and (2.2) that any two flat embeddings of a planar graph are ambient isotopic. On the other hand we have the following.

(3.1) *The graphs K_5 and $K_{3,3}$ have exactly two nonambient isotopic flat embeddings.*

Sketch of proof. Let G be $K_{3,3}$ or K_5, let e be an edge of G, and let H be $G\backslash e$. Notice that H is planar. From (2.1) it follows that if ϕ is a flat embedding of G, then there is an embedded 2-sphere $\Sigma \subseteq S^3$ with $\phi(G) \cap \Sigma = \phi(H)$. If ϕ_1 and ϕ_2 are flat embeddings of G, we may assume (by replacing ϕ_2 by an ambient isotopic embedding) that this 2-sphere Σ is the same for both ϕ_1 and ϕ_2. Now ϕ_1 is ambient isotopic to ϕ_2 if and only if $\phi_1(e)$ and $\phi_2(e)$ belong to the same component of $S^3 - \Sigma$. □

As a curiosity we deduce that a graph has a unique flat embedding if and only if it is planar.

We need the following three lemmas. We denote by $f|X$ the restriction of a mapping f to a set X.

(3.2) *Let ϕ_1, ϕ_2 be two flat embeddings of a graph G that are not ambient isotopic. Then there exists a subgraph H of G isomorphic to a subdivision of K_5 or $K_{3,3}$ for which $\phi_1|H$ and $\phi_2|H$ are not ambient isotopic.*

We denote the vertex-set and edge-set of a graph G by $V(G)$ and $E(G)$ respectively. Let G be a graph and let H_1, H_2 be subgraphs of G isomorphic to subdivisions of K_5 or $K_{3,3}$. We say that H_1 and H_2 are 1-*adjacent* if there exist $i \in \{1, 2\}$ and a path P in G such that P has only its endvertices in common with H_i and such that H_{3-i} is a subgraph of the graph obtained from H_i by adding P. We say that H_1 and H_2 are 2-*adjacent* if there are seven vertices $u_1, u_2, \ldots, u_7$ of G and thirteen paths L_{ij} of G ($1 \leq i \leq 4$ and $5 \leq j \leq 7$, or $i = 3$ and $j = 4$), such that

(i) each path L_{ij} has ends u_i, u_j,
(ii) the paths L_{ij} are mutually vertex-disjoint except for their ends,
(iii) H_1 is the union of L_{ij} for $i = 2, 3, 4$ and $j = 5, 6, 7$, and
(iv) H_2 is the union of L_{ij} for $i = 1, 3, 4$ and $j = 5, 6, 7$.

(Notice that if H_1 and H_2 are 2-adjacent then they are both isomorphic to subdivisions of $K_{3,3}$ and that L_{34} is used in neither H_1 nor H_2.) We denote by $\mathscr{K}(G)$ the simple graph with vertex-set all subgraphs of G isomorphic to subdivisions of K_5 or $K_{3,3}$ in which two distinct vertices are adjacent if they are either 1-adjacent or 2-adjacent. The following is easy to see, using (3.1).

(3.3) *Let ϕ_1, ϕ_2 be two flat embeddings of a graph G, and let H, H' be two adjacent vertices of $\mathscr{K}(G)$. If $\phi_1|H$ is ambient isotopic to $\phi_2|H$, then $\phi_1|H'$ is ambient isotopic to $\phi_2|H'$.*

The third lemma is purely graph-theoretic.

(3.4) *If G is a 4-connected graph, then $\mathscr{K}(G)$ is connected.*

We prove (3.4) in [10] by proving a stronger result, a necessary and sufficient condition for $H, H' \in V(\mathscr{K}(G))$ to belong to the same component of $\mathscr{K}(G)$ in an arbitrary graph G. The advantage of this approach is that it permits an inductive proof using the techniques of deleting and contracting edges.

Proof of (1.4). If G is planar then ϕ_1 is ambient isotopic to ϕ_2 by Mason's theorem. Otherwise there exists, by Kuratowski's theorem, a subgraph H of G isomorphic to a subdivision of K_5 or $K_{3,3}$. By replacing ϕ_2 by $-\phi_2$ we may assume by (3.1) that $\phi_1|H$ is ambient isotopic to $\phi_2|H$. From (3.3) and (3.4) we deduce that $\phi_1|H'$ is ambient isotopic to $\phi_2|H'$ for every $H' \in V(\mathscr{K}(G))$. By (3.2) ϕ_1 and ϕ_2 are ambient isotopic, as desired. □

We now state a generalization of (1.4). Let ϕ be a flat embedding of a graph G, and let $\Sigma \subseteq S^3$ be a surface homeomorphic to S^2 meeting $\phi(G)$ in a set A containing at most three points. In one of the open balls into which Σ divides S^3, say B, choose an open disk D with boundary a simple closed curve ∂D such that $A \subseteq \partial D \subseteq \Sigma$. Let ϕ' be an embedding obtained from ϕ by taking a reflection of ϕ through D in B and leaving ϕ unchanged in $\Sigma - B$. We say that ϕ' is obtained from ϕ by a 3-*switch.* The following analog of a theorem of Whitney [17] generalizes (1.4).

(3.5) *Let ϕ_1, ϕ_2 be two flat embeddings of a graph G in S^3. Then ϕ_2 can be obtained from ϕ_1 by a series of* 3-*switches.*

4. Main theorem

The difficult part of (1.2) is to show that (iii) implies (i). Let us just very briefly sketch the main idea of the proof. Suppose that G is a minor-minimal graph with no flat embedding. We first show that a $Y\Delta$-exchange preserves the property of having a flat embedding; thus we may assume that G has no triangles (and indeed has some further properties that we shall not specify here). It can be shown that G satisfies a certain weaker form of 5-connectivity. Suppose that there are two edges e, f of G so that $G\backslash e/f$ and $G/e/f$ are "Kuratowski 4-connected". (Kuratowski 4-connectivity is a slight weakening of 4-connectivity for which (1.4) still remains true.) Since G is minor-minimal with no flat embedding, there are flat embeddings ϕ_1, ϕ_2, ϕ_3 of $G\backslash e$, G/e, G/f, respectively. Since $G\backslash e/f$ and $G/e/f$ are both Kuratowski 4-connected, we can assume (by replacing ϕ_1 or ϕ_2 or both by its mirror image) that ϕ_1/f is ambient isotopic to $\phi_3\backslash e$ and that ϕ_2/f is ambient isotopic to ϕ_3/e. Now it can be argued (the details are quite complicated, see [12]) that the uncontraction of f in $\phi_1/f \simeq \phi_3\backslash e$ is the same as in $\phi_2/f \simeq \phi_3/e$. Let ϕ be obtained from ϕ_3 by doing this uncontraction; then $\phi\backslash e$ is ambient isotopic to ϕ_1 and ϕ/e is ambient isotopic to ϕ_2. Since both these embeddings are flat, ϕ is flat by (2.4), a contradiction. Thus no two such edges e, f exist. But now a purely graph-theoretic argument [11] (using the nonexistence of such edges e, f, the high connectivity of G, and that the graph obtained from G by deleting v is nonplanar for every vertex v of G) implies G has a minor in the Petersen family.

Finally we would like to mention some algorithmic aspects of flat embeddings. In [16] Scharlemann and Thompson describe an algorithm to test if a given embedding is spherical. Using their algorithm, (2.2), and (2.5), we can test if a given embedding is flat, by testing the flatness of all coforest minors. At the moment there is no known *polynomial-time* algorithm to test if an embedding of a given coforest is flat, because it includes testing if a knot is trivial. On the other hand, we can test in time $O(|V(G)|^3)$ if a given graph G has a flat embedding.

This is done by testing the absence of minors isomorphic to members of the Petersen family, using the algorithm [9] of the first two authors.

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