

MÖBIUS INVARIANCE OF KNOT ENERGY

STEVE BRYSON, MICHAEL H. FREEDMAN, ZHENG-XU HE, AND ZHENGHAN WANG

ABSTRACT. A physically natural potential energy for simple closed curves in \mathbf{R}^3 is shown to be invariant under Möbius transformations. This leads to the rapid resolution of several open problems: round circles are precisely the absolute minima for energy; there is a minimum energy threshold below which knotting cannot occur; minimizers within prime knot types exist and are regular. Finally, the number of knot types with energy less than any constant M is estimated.

Consider a rectifiable curve $\gamma(u)$ in the Euclidean 3-space \mathbf{R}^3 , where u belongs to \mathbf{R}^1 or S^1 . Define its energy by

$$E(\gamma) = \iint \left\{ \frac{1}{|\gamma(u) - \gamma(v)|^2} - \frac{1}{D(\gamma(u), \gamma(v))^2} \right\} |\dot{\gamma}(u)| |\dot{\gamma}(v)| du dv,$$

where $D(\gamma(u), \gamma(v))$ is the shortest arc distance between $\gamma(u)$ and $\gamma(v)$ on the curve. The second term of the integrand is called a regularization (see [O1–O3, FH]). It is easy to see that $E(\gamma)$ is independent of parametrization and is unchanged if γ is changed by a similarity of \mathbf{R}^3 .

Recall that the Möbius transformations of the 3-sphere $= \mathbf{R}^3 \cup \infty$ are the ten-dimensional group of angle-preserving diffeomorphisms generated by inversion in 2-spheres.

The central fact of this announcement is:

Möbius Invariant Property. *Let γ be a closed curve in \mathbf{R}^3 . If T is a Möbius transformation of $\mathbf{R}^3 \cup \infty$ and $T(\gamma)$ is contained in \mathbf{R}^3 , then $E(T(\gamma)) = E(\gamma)$. If $T(\gamma)$ passes through ∞ , the integral satisfies $E(T(\gamma)) = E(\gamma) - 4$.*

This simple fact (proved below), combined with earlier results proved in [FH], allows the rapid resolution of several open problems.

Theorem A. *Among all rectifiable loops $\gamma: S^1 \rightarrow \mathbf{R}^3$, round circles have the least energy (E (round circle) = 4) and any γ of least energy parameterizes a round circle.*

Theorem B. *If K is a smooth prime (not a connected sum) knot, then there exists a simple closed rectifiable γ_K of knot type K with $E(\gamma_K) \leq E(\gamma)$ for all rectifiable loops γ which are topologically ambient isotopic to K .*

Theorem C. *Any minimizer γ_K , as above, will enjoy some regularity. With an arc length parametrization, γ_K will be in $C^{1,1}$.*

Several results of [FH] can be improved quantitatively.

Received by the editors April 14, 1992.

1991 *Mathematics Subject Classification.* Primary 57M25, 49Q10; Secondary 53A04, 57N45, 58E30.

The first author thanks the San Diego Supercomputer Center for use of their facilities. The second and fourth authors were supported in part by NSF grant DMS-8901412. The third author was supported in part by NSF grant DMS-9006954.

Theorem D. *If γ is topologically tame, let $c([\gamma])$ denote the (topological) crossing number of the knot type. We have*

$$2\pi c([\gamma]) + 4 \leq E(\gamma).$$

(It was proved in [FH] that finite energy implies tame.)

Since an essential knot must have three or more crossings, we obtain the following

Corollary. *Any rectifiable loop with energy less than $6\pi + 4 \approx 22.84954$ is unknotted.*

Computer experiments of [A] as reported in [O3] and independently by the first author yield an essential knot (a trefoil) with energy ≈ 74 .

It may be estimated [S,T,W] that the number $K(n)$ of distinct knots of at most n crossings satisfies

$$2^n \leq K(n) \leq 2 \cdot 24^n.$$

Hence the number of knot types with representatives below a given energy threshold can also be bounded by an exponential.

Theorem E. *The number $K_e(M)$ of isomorphism classes of knots which have representatives of energy less than or equal to M is bounded by $2(24^{-4/2\pi})(24^{1/2\pi})^M \approx (0.264)(1.658)^M$. In particular, only finitely many knot types occur below any finite energy threshold.*

Note that there are competing candidates for the exponent $= -2$ in the definition of E ; for example, the Newtonian potential in \mathbf{R}^3 has exponent $= -1$. When the exponent is strictly larger than -3 , finite values are obtained for smooth simple loops. Exponents smaller or equal to -2 yield energies which blow up as a simple loop γ begins to acquire a double point, thus creating an infinite energy barrier to a change of topology. Such a barrier would not exist for the Newtonian potential. We refer to [O1–O3] for detailed discussions. Similarity and Möbius invariance are, of course, special to the exponent -2 .

Proof of Theorem A. Let T be a Möbius transformation sending a point of γ to infinity. The energy $E(T(\gamma)) \geq 0$ with equality holding iff $T(\gamma)$ is a straight line. Apply the Möbius invariant property to complete the proof. \square

Proof of Theorem B. In [FH] it is shown that for prime knot types K minimizers exist in the class of properly embedded rectifiable lines whose completion in $\mathbf{R}^3 \cup \infty$ represent K . According to the Möbius Invariance Property, such lines may be moved to a closed minimizer by any Möbius transformation T which moves the completed line off infinity. \square

Sketch of Proof of Theorem C. Let γ_K be a closed minimizer in knot type K . An inversion argument shows that, for sufficiently small $\varepsilon > 0$, if γ_K meets a closed ball of radius ε , B_ε , only in its boundary S_ε , then $\gamma_K \cap S_\varepsilon$ consists of (at most) one point. The idea is that if $\gamma_K \cap S_\varepsilon$ is disconnected, inverting an arc of $\gamma_K \setminus S_\varepsilon$ into B_ε will lower energy while preserving the knot type. Thus there is a continuous projection from the ε -neighborhood of γ_K to γ_K given by “closest point” $\pi: \mathcal{N}_\varepsilon(\gamma_K) \rightarrow \gamma_K$. We prove that the fibers $\pi^{-1}(pt)$ are all geometric planar disks of radius ε . The disjointness of these “normal” fibers

to distance ε is equivalent to the existence of a continuously turning tangent to γ_k whose generalized derivative is in L^∞ . \square

A detailed proof of Theorem C will appear elsewhere.

Proof of Theorem D. Theorem 2.5 of [FH] gives the inequality

$$c([\gamma]) \leq c(\gamma) \leq E(\gamma)/2\pi$$

for proper rectifiable lines. According to the Möbius Invariance Property, the energy will increase by exactly 4 if a Möbius transformation is used to move the line off infinity and into closed position. \square

Proof of Möbius Invariance Property. It is sufficient to consider how I , an inversion in a sphere, transforms energy. Let u be the arc length parameter of a rectifiable closed curve γ , $u \in \mathbf{R}/l\mathbf{Z}$. Let

$$(1) \quad E_\varepsilon(\gamma) = \iint_{|u-v| \geq \varepsilon} \left(\frac{1}{|\gamma(u) - \gamma(v)|^2} - \frac{1}{(D(\gamma(u), \gamma(v)))^2} \right) du dv$$

and

$$(2) \quad E_\varepsilon(I \circ \gamma) = \iint_{|u-v| \geq \varepsilon} \left(\frac{1}{|I \circ \gamma(u) - I \circ \gamma(v)|^2} - \frac{1}{(D(I \circ \gamma(u), I \circ \gamma(v)))^2} \right) \times \|I'(\gamma(u))\| \cdot \|I'(\gamma(v))\| du dv.$$

Clearly $E(\gamma) = \lim_{\varepsilon \rightarrow 0} E_\varepsilon(\gamma)$ and $E(I \circ \gamma) = \lim_{\varepsilon \rightarrow 0} E_\varepsilon(I \circ \gamma)$.

It is a short calculation (using the law of cosines) that the first terms transform correctly, i.e.,

$$\frac{\|I'(\gamma(u))\| \cdot \|I'(\gamma(v))\|}{|\gamma(u) - \gamma(v)|^2} = \frac{1}{|\gamma(u) - \gamma(v)|^2}.$$

Since u is arclength for γ , the regularization term of (1) is the elementary integral

$$(3) \quad \int_{u=0}^l \left[2 \int_{v=\varepsilon}^{l/2} \frac{1}{v^2} dv \right] du = 4 - \frac{2l}{\varepsilon}.$$

Let s be an arclength parameter for $I \circ \gamma$. Then $ds(u)/du = \|I'(\gamma(u))\|$ where $\|I'(\gamma(u))\| = f(u)$ denotes the linear expansion factor of I' . Since $\gamma(u)$ is a lipschitz function and I' is smooth, $f(u)$ is lipschitz, hence, it has a generalized derivative $f'(u) \in L^\infty$.

(4)

$$\begin{aligned} \text{regularization (2)} &= \int_{u \in \mathbf{R}/l\mathbf{Z}} \left[\int_{|v-u| \geq \varepsilon} \frac{|(I \circ \gamma)'(v)| dv}{D(I \circ \gamma(u), I \circ \gamma(v))^2} \right] |(I \circ \gamma)'(u)| du \\ &= \int_{\mathbf{R}/l\mathbf{Z}} \left[\frac{4}{L} - \frac{1}{\varepsilon_+} - \frac{1}{\varepsilon_-} \right] ds, \end{aligned}$$

where $L = \text{Length}(I(\gamma))$ and

$$\begin{aligned} \varepsilon_+ &= \varepsilon_+(u) = D((I \circ \gamma)(u), (I \circ \gamma)(u + \varepsilon)) = s(u + \varepsilon) - s(u) \\ &= \int_u^{u+\varepsilon} f(w) dw = f(u)\varepsilon + \varepsilon^2 \int_0^1 (1-t) f'(u + \varepsilon t) dt \end{aligned}$$

and

$$\varepsilon_- = \varepsilon_-(u) = D((I \circ \gamma)(u - \varepsilon), (I \circ \gamma)(u)) = f(u)\varepsilon - \varepsilon^2 \int_0^1 (1-t)f'(u - \varepsilon t) dt.$$

Since $|f'(w)|$ is uniformly bounded, we have

$$\begin{aligned} \frac{1}{\varepsilon_+} &= \frac{1}{f(u)\varepsilon} \left[\frac{1}{1 + (\varepsilon/f(u)) \int_0^1 (1-t)f'(u + \varepsilon t) dt} \right] \\ &= \frac{1}{f(u)\varepsilon} \left[1 - \frac{\varepsilon}{f(u)} \int_0^1 (1-t)f'(u + \varepsilon t) dt + \mathcal{O}(\varepsilon^2) \right] \\ &= \frac{1}{f(u)\varepsilon} - \frac{1}{f(u)^2} \int_0^1 (1-t)f'(u + \varepsilon t) dt + \mathcal{O}(\varepsilon). \end{aligned}$$

Similarly,

$$\frac{1}{\varepsilon_-} = \frac{1}{f(u)\varepsilon} + \frac{1}{f(u)^2} \int_0^1 (1-t)f'(u - \varepsilon t) dt + \mathcal{O}(\varepsilon).$$

Then by (4)

(5)

$$\begin{aligned} \text{regularization (2)} &= 4 - \int_{\mathbf{R}/l\mathbf{Z}} \frac{2}{\varepsilon} du \\ &\quad + \iint_{\mathbf{R}/l\mathbf{Z} \times [0,1]} \frac{(1-t)}{f(u)} [f'(u + \varepsilon t) - f'(u - \varepsilon t)] du dt + \mathcal{O}(\varepsilon) \\ &= 4 - \frac{2l}{\varepsilon} + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon). \end{aligned}$$

Comparing (3) and (5), we get

$$E_\varepsilon(\gamma) - E_\varepsilon(I \circ \gamma) = \mathcal{O}(\varepsilon);$$

hence, $E(\gamma) = E(I \circ \gamma)$.

For the second assertion, let l send a point of γ to infinity. In this case $L = \infty$ and, thus, the constant term 4 in (5) disappears. \square

ACKNOWLEDGMENT

The authors wish to thank Adriano Garsia and Fred Hickling for useful discussions.

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(STEVE BRYSON) COMPUTER SCIENCES CORPORATION, NASA AMES RESEARCH CENTER,
MOFFETT FIELD, CALIFORNIA 94035
E-mail address: bryson@nas.nasa.gov

(MICHAEL H. FREEDMAN, ZHENGHAN WANG) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF
CALIFORNIA AT SAN DIEGO, LA JOLLA, CALIFORNIA 92093-0112
E-mail address, M. H. Freedman: mfreedman@ucsd.edu

(ZHENG-XU HE) DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW
JERSEY 08544-1000
Current address: Department of Mathematics, Cornell University, Ithaca, New York 14853