

RELATIVE K -CYCLES AND ELLIPTIC BOUNDARY CONDITIONS

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Dedicated to Professor Zejian Jiang on his seventieth birthday

ABSTRACT. In this paper, we discuss the following conjecture raised by Baum-Douglas: For any first-order elliptic differential operator D on smooth manifold M with boundary ∂M , D possesses an elliptic boundary condition if and only if $\partial[D] = 0$ in $K_1(\partial M)$, where $[D]$ is the relative K -cycle in $K_0(M, \partial M)$ corresponding to D . We prove the “if” part of this conjecture for $\dim(M) \neq 4, 5, 6, 7$ and the “only if” part of the conjecture for arbitrary dimension.

First we fix some notation. M is a compact oriented smooth manifold with smooth boundary ∂M . We always suppose that M is embedded in some compact smooth manifold \widetilde{M} without boundary of the same dimension (e.g., \widetilde{M} can be taken as double of M). We denote $\overset{\circ}{M} = M \setminus \partial M$. Furthermore, we assume that E_0 and E_1 (in fact, all the vector bundles in this paper) are smooth complex Hermitian vector bundles over M and that $D : C^\infty(E_0) \rightarrow C^\infty(E_1)$ is a first-order elliptic differential operator from smooth sections of E_0 to that of E_1 . By $H^s(M, E_i)$ and $H^s(\partial M, E_i)$ we shall denote the Sobolev spaces of sections of E_i and $E_i|_{\partial M}$ with respect to fixed smooth measures on M and ∂M , respectively.

The elliptic boundary value problem (an elliptic operator with an elliptic boundary condition) has been studied for a long time. As noted in [1, 5, 6] and other references, there exist topological obstructions to impose an elliptic boundary condition on the above D . A fundamental problem is to find all such obstructions. Baum, Douglas, and Taylor constructed a relative K -cycle $[D] \in K_0(M, \partial M) \cong KK(C_0(\overset{\circ}{M}), \mathbb{C})$ (here $C_0(\overset{\circ}{M})$ is the algebra of continuous functions on M which vanish on ∂M) corresponding to D (see [2–4] for details). From the definition of relative K -homology group $K_0(M, \partial M)$ given by Baum, Douglas, and Taylor, the boundary map $\partial : K_0(M, \partial M) \rightarrow K_1(\partial M)$ is very concrete [2–4]. Also Baum and Douglas conjectured that the only obstruction for D possessing elliptic boundary conditions is that $\partial[D] \neq 0$. More precisely, the following conjecture first appeared in [2] in a closely related form.

Conjecture. *There exist a vector bundle E_2 over ∂M and a zeroth-order pseudo-differential operator B defined from $C^\infty(\partial M, E_0)$ to $C^\infty(\partial M, E_2)$ such that*

$$\begin{pmatrix} D \\ B \circ \gamma \end{pmatrix} : H^1(M, E_0) \longrightarrow \begin{matrix} H^0(M, E_1) \\ \oplus \\ H^{1/2}(\partial M, E_2) \end{matrix}$$

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is Fredholm if and only if $\partial[D] = 0$ in $K_1(\partial M)$, where $\gamma : H^1(M, E_0) \rightarrow H^{1/2}(\partial M, E_0)$ is the trace map.

Remark 1. Let D be as above with principal symbol $p(x, \xi)$. A zeroth-order pseudo-differential operator B with principal symbol $b(x, \xi)$ from $C^\infty(\partial M, E_0)$ to $C^\infty(\partial M, E_2)$ is said to be elliptic to D (see [6], p. 233) if, for every $x \in \partial M$ and $\xi \in T_x^*(\partial M)$, the map $M_{x,\xi}^+ \ni u \rightarrow b(x, \xi)u(0) \in (E_2)_x$ is bijective, where $T_x^*(\partial M)$ and $(E_2)_x$ are the fibres at the point x of the cotangent bundle $T^*\partial M$ and the bundle E_2 , and, furthermore, $M_{x,\xi}^+$ is the set of all $u \in C^\infty(\mathbb{R}, (E_0)_t)$ with $p(x, \xi - i\frac{d}{dt} \cdot n_x)u(t) = 0$ (n_x is the interior conormal vector of M at x) which are bounded on \mathbb{R}_+ . If B is elliptic to D , then the above $(\begin{smallmatrix} D \\ B \circ \gamma \end{smallmatrix})$ is Fredholm. Such a system $(\begin{smallmatrix} D \\ B \circ \gamma \end{smallmatrix})$ is often called an elliptic boundary value problem or an elliptic operator with an elliptic boundary condition; meanwhile, D is also said to possess an elliptic boundary condition.

Remark 2. Although the above elliptic boundary condition is used in most references, the original form of the conjecture in [2] is in a slightly different form from the above. In [2] the operator for the boundary condition is of the form $\gamma \circ B$, where B is a zeroth-order pseudo-differential operator from E_0 to a smooth vector bundle over a neighborhood of M . The reason we use a slightly different form of the conjecture is as follows: for general zeroth-order pseudo-differential operator B defined on \widetilde{M} , there is no canonical way to restrict B to M as an operator $B_M : H^s(M) \rightarrow H^s(M)$ when $s > 0$. So one needs to put some restriction on B . One of the natural restrictions is that B has the transmission property with respect to ∂M (see [5]). We also prove our theorem for this kind of boundary condition (see Theorem 1). It must be pointed out that the existence of a boundary condition of type $B \circ \gamma$ implies the existence of that of type $\gamma \circ B$.

In this paper, we prove the “only if” part of the conjecture which can be thought of as a generalization of Corollary 4.2 in [4] (there B is a differential operator). Conversely, we prove that if $\dim(M) \neq 4, 5, 6, 7$ and $\partial[D] = 0$, then D possesses an elliptic boundary condition as in Remark 1. Hence the “if” part of the conjecture has been proved for $\dim(M) \neq 4, 5, 6, 7$. The cases of $\dim(M)$ being equal to 4, 5, 6, or 7 are still open, but we prove a theorem which can be thought of as the “if” part of the conjecture in the sense of stabilization in K -homology group for arbitrary dimension. Our results will be useful for constructing absolute K -cycles in $K_0(M)$ which are preimages of $[D] \in K_0(M, \partial M)$ under the canonical map from $K_0(M)$ to $K_0(M, \partial M)$ when $\partial[D] = 0$.

Our main results are the following:

Theorem 1. (“only if” part) $\partial[D] = 0$ if one of the following is true:

(i) There exist a smooth vector bundle E_2 over ∂M and a zeroth-order pseudo-differential operator B from $E_0|_{\partial M}$ to E_2 such that $(\begin{smallmatrix} D \\ B \circ \gamma \end{smallmatrix})$ in the conjecture is Fredholm.

(ii) There exist a bundle E_2 over a neighborhood of M in \widetilde{M} and a zeroth-order pseudo-differential operator B with transmission property with respect to

∂M from E_0 to E_2 such that

$$\begin{pmatrix} D \\ \gamma \circ B \end{pmatrix} : H^1(M, E_0) \longrightarrow \begin{matrix} H^0(M, E_1) \\ \oplus \\ H^{1/2}(\partial M, E_2) \end{matrix}$$

is Fredholm.

Theorem 2. (“if” part) *If $\partial[D] = 0$, then there exists a first-order elliptic differential operator D_1 acting on smooth vector bundles over M with $[D_1] = 0$ in $K_0(M, \partial M)$ such that $D \oplus D_1$ possesses an elliptic boundary condition as in Remark 1, and, furthermore, if $\dim(M) \neq 4, 5, 6, 7$, then D itself possesses an elliptic boundary condition.*

The main idea of the proof of Theorem 1 is to construct an intertwining between $\partial[D]$ and a trivial element in $K_1(\partial M)$. In the proof, we use Calderon projection, functional calculus of pseudo-differential operators (including Boutet de Monvel type operators), and the techniques in the proof of Proposition 4.5 of [4].

The proof of Theorem 2 makes use of two key lemmas (see below).

Let $ST^*\partial M$ be the unit sphere bundle of $T^*\partial M$ over ∂M and $\pi : ST^*\partial M \rightarrow \partial M$ be the canonical projection map. Let $\tilde{E}_0 = \pi^*(E_0|_{\partial M})$ be the bundle over $ST^*\partial M$. We write the principal symbol of D , in a coordinate neighborhood U of $x \in \partial M$, as

$$p(x, x_n, \xi, \xi_n) = \sum_{j=1}^{n-1} p_j(x, x_n)\xi_j + p_n(x, x_n)\xi_n,$$

where x_n is the coordinate for the normal direction of ∂M . We define

$$\tau(x, \xi) = ip_n^{-1}(x, 0) \sum_{j=1}^{n-1} p_j(x, 0)\xi_j$$

for $x \in \partial M$ and $\xi \in ST^*\partial M$. Then $\tau(x, \xi)$ is a map from a fibre of E_0 into itself and has no purely imaginary eigenvalue. Let V_{\pm} be the subbundle of \tilde{E}_0 over $ST^*\partial M$ corresponding to the span of the generalized eigenvectors of $\tau(x, \xi)$ corresponding to the eigenvalues with positive/negative real parts.

Lemma 1. $\partial[D] = 0$ if and only if $[V_+] \in \pi^*K^0(\partial M) \subset K^0(ST^*\partial M)$.

Lemma 2. *If E_0 and E_1 are vector bundles over M which allow a first-order elliptic differential operator D to act from one to the other, and if $\dim(M) = n$, then*

- (i) $f \dim(E_0) = f \dim E_1 \geq 2^{[(n-1)/2]}$;
- (ii) $f \dim(E_0) = f \dim E_1 \geq 2^{[(n-1)/2]+1}$ provided n is even and $\partial[D] = 0$, where $f \dim$ denotes dimension of each fibre of the vector bundles.

The Proof of Theorem 2. By Lemma 1, if $\partial[D] = 0$, one has

$$[V_+] \in \pi^*K^0(\partial M) \subset K^0(ST^*\partial M).$$

By Lemma 2, $f \dim(V_+) = \frac{f \dim(E_0)}{2} \geq 2^{[n/2]-1}$. Therefore, $f \dim(V_+) \geq n - 1 > \frac{\dim(ST^*\partial M)}{2} = \frac{2n-3}{2}$, whenever $\dim(M) \geq 8$. Hence there exists a complex

vector bundle E_2 over ∂M , such that $V_+ \cong \pi^*E_2$. This is also true for $\dim(M) \leq 3$, since the collection of complex vector bundles over $ST^*\partial M$ ($\dim(ST^*\partial M) \leq 3$) has property of cancellation.

Let ψ be the bundle isomorphism

$$\begin{array}{ccc} V_+ & \xrightarrow{\psi} & \pi^*E_2 \\ \downarrow & & \downarrow \\ ST^*\partial M & \longrightarrow & ST^*\partial M \end{array}$$

For any $(x, \xi) \in ST^*\partial M$, let $b(x, \xi)$ be the bundle map defined by

$$\begin{array}{ccccc} E_0 & \xrightarrow{\text{project to}} & V_+ & \xrightarrow{\psi} & E_2 \\ \downarrow & & & & \downarrow \\ ST^*\partial M & \longrightarrow & ST^*\partial M & \longrightarrow & ST^*\partial M \end{array}$$

Furthermore, let $B : E_0|_{\partial M} \longrightarrow E_2$ be the zeroth-order pseudo-differential operator with symbol $b(x, \xi)$. It follows that B is elliptic to D (see [6]). \square

Example 1. If M is a $spin^c$ manifold with smooth boundary and D is the Dirac operator over M , then it is computed in [4] that $\partial[D] \neq 0$. Hence D possesses no elliptic boundary condition (even possesses no boundary condition as in the conjecture).

Example 2. For any D , let D^* be the formal adjoint of D . It is easy to prove that $[D] = -[D^*]$ in $K_0(M, \partial M)$; hence, $\partial[D \oplus D^*] = 0$. It follows that $D \oplus D^*$ possesses an elliptic boundary condition provided $\dim(M) \neq 4$ or 6 . (It should be noted that we only need to exclude the manifolds with dimension 4 and 6 here, since the dimension of the bundle on which $D \oplus D^*$ acts is twice the dimension of the bundle on which D acts.)

The details of the proofs will appear elsewhere.

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REFERENCES

1. M. Atiyah, V. Patodi, and I. Singer, *Spectral asymmetry and riemannian geometry*. I, II, III, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69; **78** (1975), 405–632; **79** (1976), 71–99.
2. P. Baum and R. Douglas, *Index theory, bordism, and K-homology*, Operator Algebras and K-Theory (R. G. Douglas and C. Schochet, eds.), Contemp. Math., vol. 10, Amer. Math. Soc., Providence, RI, 1982, pp. 1–31.
3. ———, *Relative K-homology and C*-algebra*, manuscript.
4. P. Baum, R. Douglas, and M. Taylor, *Cycles and relative cycles in analytic K-homology*, J. Differential Geom. **30** (1989), 761–804.

5. L. Boutet de Monvel, *Boundary problems for pseudodifferential operators*, Acta Math. **126** (1971), 11–51.
6. L. Hörmander, *The analysis of linear partial differential operators. III*, Springer, New York, 1985.

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