BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 1, January 1993

0273-0979/93 $1.00 + $.25 per page

The geometry of supermanifolds, by Claudio Bartocci, Ugo Bruzzo, and Daniel Hernández-Ruipérez. Kluwer Academic Publ., Dordrecht 1991, 242 pp., $77.00. ISBN 0-7923-1440-9

Over the last two decades supersymmetric quantum theories have been studied intensively in the belief that such theories may play a part in a unified theory of the fundamental forces. This work has inspired the study of a new class of mathematical structures, which are extensions of the conventional mathematical structures used in quantum physics, such as Lie groups and manifolds, to include (in some sense) commuting as well as anticommuting variables. The prefix 'super' (which is technical rather than descriptive) is usually used to denote such extensions, in particular, a supermanifold is a manifold modeled on a space with some commuting and some anticommuting coordinates.

Anticommuting variables were first introduced into physics by Martin [12], with the aim of extending Feynman path integrals to fermions. The earliest work in supermathematics, some of which predates the discovery of supersymmetry in physics, was motivated by both the intrinsic mathematical interest of the topic and the desire to extend to fermions standard analytic methods used for handling the quantization of bosons. The extensive work of Berezin [2, 3] and the work of Kostant [10] provided an analytic and geometric framework for the use of anticommuting variables. Supersymmetric theories involve transformations which intertwine bose and fermi fields and thus require a formulation which treats bosons and fermions on an equal footing; moreover, the formalism requires that classical fermions be regarded as anticommuting quantities. These features led to a greatly intensified study of supermathematics after the discovery of supersymmetry in physics.

A brief informal description of a simple supersymmetric model will now be given in order to show how the need for supergeometry emerges. (For a reader unfamiliar with quantum field theory, it is the transformations (2) and the algebra (4) that are important.) A more complete account of supersymmetry may be found in many places, such as in the work of West [19].

The simplest supersymmetric model (in a four-dimensional spacetime) was discovered by Wess and Zumino in 1974 [18]. The Wess-Zumino model has fields A, B, χ, F, and G, where A, B, F, and G are bosonic fields and χ is fermionic, with A and B scalar fields, F and G auxiliary (nonpropagating) fields, and χ a Majorana spinor with components χ_α, $\alpha = 1, \dots, 4$. The action of the free theory is

$$(1) \qquad S = \int d^4x \left(-\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\chi}\gamma^\mu \partial_\mu \chi + \frac{1}{2}F^2 + \frac{1}{2}G^2 \right),$$

where γ^μ are Dirac matrices. (The conventions of [19] are used.) This action

is invariant under the set of infinitesimal transformations of the fields

$$
\begin{aligned}
&\delta_\varepsilon A = \bar{\varepsilon}\chi\,,\\
&\delta_\varepsilon B = i\bar{\varepsilon}\gamma_5\chi\,,\\
&\delta_\varepsilon \chi = (F + i\gamma_5 G + \gamma^\mu\partial_\mu(A + i\gamma_5 B))\varepsilon\,,\\
&\delta_\varepsilon F = \bar{\varepsilon}\partial_\mu\chi\,,\\
&\delta_\varepsilon G = i\bar{\varepsilon}\gamma_5\gamma^\mu\partial_\mu\chi
\end{aligned}
\tag{2}
$$

provided that the parameter ε and the field χ anticommute with one another. Transformations of this nature, which interchange bosonic and fermionic fields, are known as supersymmetry transformations. The commutator of two such transformations is a translation

$$
\delta_{\varepsilon_1}\delta_{\varepsilon_2} - \delta_{\varepsilon_2}\delta_{\varepsilon_1} = 2\overline{\varepsilon_2}\gamma^\mu\varepsilon_1\partial_\mu
\tag{3}
$$

(provided that ε_1, ε_2, and χ are all mutually anticommuting), but the parameter $2\overline{\varepsilon_2}\gamma^\mu\varepsilon_1$ of the translation is not simply a real or complex number. Now direct calculation shows that translations commute with supersymmetry transformations, and thus a closed algebra of transformations is obtained. Suppose that $\delta_\varepsilon = \bar{\varepsilon}Q$, so that Q_α, $\alpha = 1, \dots, 4$, denote the generators of the supersymmetry transformations; also let P_μ, $\mu = 0, \dots, 3$, denote the translation generators. Then the generators satisfy the anticommutation and commutation relations

$$
\begin{aligned}
Q_\alpha Q_\beta + Q_\beta Q_\alpha &= 2(\gamma^\mu_{\alpha\beta})P_\mu\,,\\
Q_\alpha P_\mu - P_\mu Q_\alpha &= 0\,,\\
P_\mu P_\nu - P_\nu P_\mu &= 0\,,
\end{aligned}
\tag{4}
$$

which provide an example of a super Lie algebra. The infinitesimal transformations (2) with closing algebra (4) suggest the need for some sort of group with elements of the form (loosely speaking) $\exp(\sum_{\mu=0}^{3} x^\mu P_\mu + \sum_{\alpha=1}^{4}\theta^\alpha Q_\alpha)$ where the parameters x^μ are commuting and the parameters θ^α are anticommuting. Such a group is known as a super Lie group and is the first example of the new type of mathematics that arises out of supersymmetry. In the physics literature such groups were regarded as matrix groups, with some of the matrix entries commuting and some anticommuting; considerable progress was made in these heuristic terms. More mathematically, Berezin and Kac developed a theory of formal groups with commuting and anticommuting parameters [4], while Kostant [10] and Rogers [14] gave definitions of super Lie groups in terms of supermanifolds, which are considered below.

A crucial mathematical innovation in supersymmetry was the introduction of superspace by Salam and Strathdee [16] and by Volkov and Akulov [17]. Superspace is useful both technically, for generating representations of the supersymmetry algebra (4), and conceptually, in providing a geometric setting in which the supersymmetry transformations are generalised translations. For the supersymmetric model described above, which is defined on the four-dimensional Minkowski spacetime of special relativity, the appropriate superspace is (informally) a space parametrised by four commuting variables x^μ, $\mu = 0, \dots, 3$, and four anticommuting variables θ^α, $\alpha = 1, \dots, 4$. The supersymmetry

algebra (4) is realised on this space by setting

$$\begin{aligned} \bar{\varepsilon}Q(x^\mu) &= x^\mu - \bar{\varepsilon}\gamma^\mu\theta\,, \\ \bar{\varepsilon}Q(\theta^\alpha) &= \theta^\alpha + \varepsilon^\alpha\,, \end{aligned} \tag{5}$$

as can be checked by explicit calculation. Representations of the supersymmetry algebra can then be obtained from functions $V(x\,,\theta)$ defined on superspace; the key observation is that, because of the anticommuting nature of the θ variables, such functions have finite Taylor expansions of the form

$$\begin{aligned} V(x\,,\theta) = V(x) &+ \sum_\alpha V_\alpha(x)\theta^\alpha + \sum_{\alpha<\beta} V_{\alpha\beta}\theta^\alpha\theta^\beta \\ &+ \sum_{\alpha<\beta<\gamma} V_{\alpha\beta\gamma}(x)\theta^\alpha\theta^\beta\theta^\gamma + V_4(x)\theta^1\theta^2\theta^3\theta^4. \end{aligned} \tag{6}$$

It is from the coefficient functions $V(x)$, $V_\alpha(x)$, ... that the Minkowski space fields are recovered. The supersymmetry algebra can be represented on superfields by setting

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - (\gamma^\mu\theta)_\alpha\frac{\partial}{\partial x^\mu}\,, \qquad P_\mu = \frac{\partial}{\partial x^\mu}. \tag{7}$$

Using Berezin's theory of integration of anticommuting variables [2], invariants can be constructed from suitable combinations of superfields.

In 1976 the first of a number of supergravity theories, combining supersymmetry and gravity, was discovered [8, 6]; this led to the consideration of superspaces which were extensions of general pseudo-Riemannian and Riemannian manifolds to include anticommuting coordinates, and thus the concept of supermanifold was required; tensor calculus, integration theory, and all the machinery of differential geometry were needed for the full development of superspace techniques for supergravity and, later, superstrings.

Two different but related approaches to the rigorous formulation of supermanifold theory have been developed. One is the sheaf-theoretic approach, in which it is the ring of functions on a manifold, rather than the manifold itself, which is extended to include anticommuting variables. In this approach, pioneered by Berezin and Leĭtes [5] and Kostant [10], an $(m\,,n)$-dimensional real supermanifold is a pair $(M\,,A)$ where M is an m-dimensional conventional manifold and A is a sheaf of graded algebras over M which locally takes the form

$$A(U) \cong C^\infty(U)\otimes\Lambda(\mathbb{R}^n)\,, \qquad U\subset M \tag{8}$$

(with $\Lambda(\mathbb{R}^n)$ denoting the exterior algebra over $\mathbb{R}^n$). Thus, if $f\in A(U)$, f may be expressed as

$$f = f_\varnothing + \sum_j f_j\xi^j + \sum_{j<k} f_{jk}\xi^j\xi^k + \cdots\,, \tag{9}$$

where the coefficients $f_\varnothing$, f_j, ... are elements of $C^\infty(U)$ and ξ_j, $j=1,\dots,n$, are generators of $\Lambda(\mathbb{R}^n)$. (Complex analytic supermanifolds may be defined in an analogous manner.) Elements of $A(U)$ superficially resemble superfields, but there are differences, in that first the coefficient functions such as f_j are all commuting and second the formalism does not include constant anticommuting elements and so does not allow supersymmetry transformations of superspace.

This sheaf-theoretic approach to supermanifolds is elegant and economical but needs extension to incorporate the mathematical structures used in supersymmetric physics.

The alternative approach to supermanifolds is to mimic the definition of conventional real or complex manifolds, but with the coordinate space $\mathbb{R}^m$ or $\mathbb{C}^m$ replaced by a space $\mathbb{B}^{m,n}$ parametrised by m commuting and n anticommuting variables. This geometric approach to supermanifolds grew directly out of the superspace constructions of supersymmetric physics. In this approach a supermanifold is a topological space N together with an atlas of charts $\{(U_\alpha, \psi_\alpha)\}$ where the U_α cover N and each coordinate function ψ_α maps the corresponding U_α onto an open subset of $\mathbb{B}^{m,n}$, with the transition functions $\psi_\alpha \circ \psi_\beta^{-1}$ between overlapping coordinate patches required to be supersmooth in some sense. Thus the geometric approach requires the specification of the topological space $\mathbb{B}^{m,n}$, with appropriate algebraic properties, and some notion of a supersmooth function of $\mathbb{B}^{m,n}$ into itself. A number of different versions of $\mathbb{B}^{m,n}$ have appeared in the literature; generally $\mathbb{B}^{m,n}$ is seen as the Cartesian product of m copies of the even part of a Grassmann algebra $\mathbb{B}$ and n copies of the odd part, but the Grassmann algebra may be finite dimensional or infinite dimensional and may be topologised as a Banach space [13], a Frechet space [9], or, with a very coarse, non-Hausdorff topology introduced by DeWitt [7]. (DeWitt's topology reduces the class of possible supermanifolds in a useful way. Open sets in $\mathbb{B}^{m,n}$ in the DeWitt topology are in one-to-one correspondence with open sets in $\mathbb{R}^m$ in the usual topology.) The different choices of $\mathbb{B}^{m,n}$ make the supermanifold literature very confusing; however, one can identify generic features of supermanifolds which are obtained with all reasonable choices of $\mathbb{B}^{m,n}$. Finite-dimensional Grassmann algebras are easier to handle but lead to ambiguities in derivatives with respect to the anticommuting coordinates; these can be circumvented, or avoided by using infinite-dimensional Grassmann algebras, but this of course can lead to analytical problems. A useful axiomatisation has been developed by Rothstein [15]. Supersmooth functions are defined to be functions of the form

$$f(x, \theta) = f_\varnothing(x) + \sum_i f_i(x)\theta^i + \sum_{i<j} f_{ij}(x)\theta^i\theta^j + \cdots \tag{10}$$

where the coefficient functions $f_\varnothing, f_i, \ldots$ are smooth functions of $\mathbb{R}^m$ into $\mathbb{B}$ (or a subalgebra of $\mathbb{B}$) extended from $\mathbb{R}^m$ (regarded as a subset of $\mathbb{B}^{m,0}$ in the natural way) to all of $\mathbb{B}^{m,0}$ by Taylor expansion. Such functions closely resemble the formal algebras used in the sheaf-theoretic approach, indeed the two approaches are closely related, as is demonstrated by Batchelor [1] and Rogers [13]. The supersmooth functions (10), however, resemble the physicist's superfields more closely than the formal functions of the sheaf-theoretic approach, because the coefficient functions $f_\varnothing, \ldots$ take values in $\mathbb{B}$ or (in order to obtain well-defined derivatives with respect to odd variables when $\mathbb{B}$ is finite-dimensional) a Grassmann subalgebra of $\mathbb{B}$. This allows a rigorous implementation of superspace techniques. Thus the geometric approach is more cumbersome than the algebraic in that one has to introduce the auxiliary Grassmann algebra $\mathbb{B}$, but it is more directly adapted to applications to supersymmetric physics.

Many standard constructions from classical differential geometry can be adapted to supermanifolds in a straightforward way. The starting point (in

both approaches) is the ring of smooth functions, which acquires a Z_2-grading from the corresponding grading of the Grassmann algebra. A vector field is then defined to be an endomorphism D of the function ring which satisfies the graded Leibnitz rule

$$Dfg = (Df)g + (-1)^{|D||f|} fDg \tag{11}$$

for all supersmooth functions f and g. ($|f|$ and $|D|$ denote the Z_2 gradings of f and D, respectively.) Further tensors, and various derivatives, can be defined much as in the classical case, provided that a factor of (-1) is introduced whenever the order of two odd factors is changed, as in the graded Leibnitz rule (11); however, there is no such straightforward extension of integration theory, which has to be radically modified along lines developed initially by Berezin [2]. An account of integration theory on compact supermanifolds is included in the useful review article of Leĭtes [11].

The book under review gives a carefully constructed, accurate, and readable account of the basic definitions of supermanifolds, together with detailed descriptions of those areas of the theory in which the authors have particularly worked, notably, supervector bundles and cohomology.

The book begins with two excellent sections on foundations, covering the basics of graded algebras and sheaf theory. This is followed by an introduction to both approaches to supermanifold theory, together with a detailed analysis of the relation between the various definitions. The authors then define a G-supermanifold, which is the definition of supermanifold principally used in the remainder of the book. A G-supermanifold is a pair (N, A) with N a geometric supermanifold modeled on a finite-dimensional Grassmann algebra $\mathbb{B}$ and A the tensor product of the sheaf of supersmooth functions with $\mathbb{B}$. This definition is made so that certain key features of classical vector bundle theory are retained; in particular, one may recover the fibre of a bundle as the space of sections at a point modulo those sections which are zero at that point, which is only possible if sheaves of functions (and cross-sections) are modified by the tensor product.

The later sections of the book are more specialised and are largely based on the authors' own work. A section on the cohomology of supermanifolds introduces cohomology groups which are G-supermanifold invariants but not differential or topological invariants; various De Rham theorems are proved, and all DeWitt supermanifolds are shown to have acyclic structure sheaves.

Supervector bundles are then introduced, and it is shown that the geometric definition corresponds exactly with the algebraic definition of a supervector bundle as a locally free A module. Connections are defined, and conditions for their existence are established. The Chern classes of a supervector bundle are defined in terms of the obstruction class of the tautological bundle of the projectivization of the bundle. Where connections exist, a representation of the Chern classes in terms of the curvature is derived.

The final chapter gives a careful discussion of super Lie groups and super principal bundles.

This is an extremely well-written book that explains complex ideas very clearly and makes a significant contribution to the literature on supermanifolds. It is recommended as providing a particularly lucid introduction to the basic theory of supermanifolds and a good and novel analysis of the cohomological

aspects of supermanifolds and supervector bundles.

One or two minor criticisms one might make are that the important topic of integration is omitted and that little attention is given to the particular structures used in applications to supersymmetric physics. A useful notational innovation is the use (in addition to the black square at the end of a proof) of a black triangle at the end of a remark or example, so that it is immediately clear where the main text resumes.

A major criticism is the extraordinarily high price.

References

1. M. Batchelor, *Two approaches to supermanifolds*, Trans. Amer. Math. Soc. **258** (1980), 257–270.
2. F. A. Berezin, *The method of second quantization*, Academic Press, New York, 1966.
3. ———, *Introduction to superanalysis*, Reidel, Dordrecht, 1987.
4. F. A. Berezin and G. I. Kac, *Lie groups with commuting and anticommuting parameters*, Math. USSR Sb. **11** (1970), 311.
5. F. A. Berezin and D. A. Leĭtes, *Supermanifolds*, Soviet Math. Dokl. **6** (1976), 1218–1222.
6. S. Deser and B. Zumino, *Consistent supergravity*, Phys. Lett. B **62** (1976), 335.
7. B. S. DeWitt, *Supermanifolds*, Cambridge Univ. Press, London and New York, 1984.
8. S. Ferrera, D. Freedman, and P. van Nieuwenhuizen, *Progress towards a theory of supergravity*, Phys. Rev. D (3) **13** (1976), 3214.
9. A. Inoue and Y. Maeda, *Foundations of calculus on super euclidean space $R^{m|n}$ based on a Fréchet-Grassmann algebra*, Kodai Math. J. **14** (1991), 72–112.
10. B. Kostant, *Graded manifolds, graded Lie theory and prequantization*, Differential Geometric Methods in Mathematical Physics, Lecture Notes in Math., vol. 570, Springer-Verlag, New York, 1977.
11. D. A. Leĭtes, *Introduction to the theory of supermanifolds*, Russian Math. Surv. **35** (1980), 1–64.
12. J. Martin, *The feynman principle for a fermi system*, Proc. Roy. Soc. London Ser. A, **251** (1959), 543–549.
13. A. Rogers, *A global theory of supermanifolds*, J. Math. Phys. **21** (1980), 1352–1365.
14. ———, *Super Lie groups: global topology and local structure*, J. Math. Phys. **22** (1981), 939–945.
15. M. Rothstein, *The axioms of supermanifolds and a new structure arising from them*, Trans. Amer. Math. Soc. **297** (1986), 159–180.
16. Abdus Salam and J. Strathdee, *Supergauge transformations*, Nuclear Phys. B Proc. Suppl. (1974).
17. D. V. Volkov and V. P. Akulov, *Is the neutrino a Goldstone particle?*, Phys. Lett B **46** (1973), 109–110.
18. J. Wess and B. Zumino, *Supergauge transformations in four dimensions*, Nuclear Phys. B **70** (1974), 39.
19. Peter West, *Introduction to supersymmetry and supergravity*, World Scientific, Singapore, 1990.

Alice Rogers
King's College London, University of London
E-mail address: F. A. ROGERS @UK.AC.KCL.CC.OAK