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The theory of subnormal operators, by John Conway. Math. Surv. Mono., vol. 36, Amer. Math. Soc., Providence, RI, 1991, xv + 436 pp., $64.00. ISBN 0-8218-1536-9

From the vantage point of an outsider, it seems there are two kinds of operator theorists; one deals with operators in bulk, the other deals with them one by one. This book is written by a practitioner of the second school. The aim is to provide an exposition of relevant parts of rational approximation theory and to describe how the technology can be applied to subnormal operators.

Selfadjoint and normal operators were the first classes of Hilbert-space operators to be analyzed. The prototype is multiplication by the coordinate function z on $L^2(\mu)$, where μ is a compactly supported measure on the complex plane. If we take μ to be the arc-length measure on the unit circle, we obtain the shift operator on two-tailed square-summable sequences. The spectral theorem asserts that any normal operator can be represented as a direct sum of such multiplication operators on L^2-spaces. The theory has the flavor of measure and integration.

An operator S is defined to be *subnormal* if it is the restriction of a normal operator to an invariant subspace. These operators, introduced in 1950 by P. Halmos, have as prototype the multiplication by z on the closure $P^2(\mu)$ in $L^2(\mu)$ of the analytic polynomials. In the case of arc-length measure we obtain multiplication by z on the Hardy space $H^2(d\theta)$, which is equivalent to the shift operator on one-sided square-summable sequences. Now in addition to measure theory there is a strong taste of function theory. One asks, when are the polynomials in z dense in $L^2(\mu)$? If they are not dense, can the defect be accounted for by the analyticity of functions in $P^2(\mu)$ on some nonempty open subset of the plane?

Function theory enters the picture through the functional calculus. Let K be a compact subset of the complex plane containing the spectrum $\sigma(S)$ of S. If f is a rational function with poles off K, then $f(S)$ is defined, and the operator norm of $f(S)$ is dominated by the supremum norm of f over K. Thus we obtain an algebra homomorphism of the uniform algebra $R(K)$ into $\mathscr{B}(\mathscr{H})$. The latter space is a dual space, and if μ is a measure on K with sufficiently ample support, the operator calculus extends to a weak-star continuous homomorphism from the weak-star closure $R^\infty(K, \mu)$ of $R(K)$ in $L^\infty(\mu)$ into $\mathscr{B}(\mathscr{H})$. For $K = \sigma(S)$ and special μ, we have even an isometric weak-star homeomorphism. Information about $R^\infty(K, \mu)$ yields information about S via this functional calculus. The functional calculus also extends to the bidual $R(K)^{**}$, which is an inverse limit of the spaces $R^\infty(K, \mu)$.

The problem of uniform approximation by analytic polynomials was solved in 1953 by S. N. Mergelyan. The uniform limits on K of polynomials in z are precisely the functions in $C(K)$ which extend continuously to be analytic on the interior of the polynomial hull of K (the union of K and the bounded components of the complement of K). Mergelyan's proof was completely constructive. He used the Cauchy-Green formula to split the singularities into bite-sized chunks. This localization technique and the Cauchy transform permeate the constructive side of the theory. In 1963 a semiabstract proof of Mergelyan's theorem was obtained through the efforts of E. Bishop, I. Glicksberg, and J. Wermer. Concrete function theory still played an important role. The abstract theory kicks in once one knows that $P(K)$ is a Dirichlet algebra, and this requires an (easier) approximation theorem for harmonic functions.

The L^2-approximation problem remains a difficult nut to crack, with complete results only in very special cases. On the other hand, the weak-star approximation problem turned out to be more tractable. In 1972, through a beautiful analysis, D. Sarason obtained a decomposition of $P^\infty(\mu)$ as a direct sum of $L^\infty(\mu_s)$ and $P^\infty(\mu_a)$, where the latter summand is isometric and weak-star homeomorphic to the algebra $H^\infty(U)$ of bounded analytic functions on a certain open set U. Sarason's proof is constructive in nature. He constructs an increasing chain of Dirichlet algebras $R(K_\alpha)$, indexed by the ordinals, such that functions in a dense subset of each can be approximated appropriately by bounded sequences of functions from the predecessors. The procedure terminates long before the first uncountable ordinal is reached, and U is the interior of the final (smallest) K_α. Sarason's analysis had a substantial impact on the study of subnormal operators. The functional calculus that was obtained played a role in S. Brown's proof in 1978 of the existence of invariant subspaces for subnormal operators. In turn, this theorem attracted a lot of attention to the theory. In 1981 Conway published a research monograph in the red Pitman series that described this theory and culminated in Sarason's analysis of $P^\infty(\mu)$ and Brown's invariant subspace theorem; see the 1983 review by P. Muhly in this journal.

The next natural step was to apply the tools of rational approximation theory, specifically properties of $R^\infty(K, \mu)$, to obtain information on subnormal operators. The problem of uniform approximation by rational functions had been solved in 1967 by A. G. Vitushkin in terms of analytic capacities. In 1972, A. M. Davie obtained a striking result, a decomposition of the bidual $R(K)^{**}$ as a direct sum of an L^∞-space and an algebra isometric and weak-star homeomorphic to $R^\infty(K, \lambda_Q)$, where λ_Q is the area measure restricted to the set Q of nonpeak points of $R(K)$. The set Q has full area density at each of its points, and in some respects it behaves as a finely open set does with respect to bounded harmonic functions. Intuitively the algebra $R^\infty(K, \lambda_Q)$ can be thought of as an algebra of bounded analytic functions on Q. In the case $R(K)$ is a Dirichlet algebra, Q is just the interior of K.

The next step was taken in 1974 by J. Chaumat, who gave a similar description of $R^\infty(K, \mu)$ for an arbitrary measure μ on K. The set playing the role of Q, denoted by $E(\mu)$, consists of the points $z \in K$ for which the evaluation functional is weak-star continuous on $R(K)$ in the weak-star topology of L^∞ of the measure μ deprived of its mass at $\{z\}$. Again $E(\mu)$ has full area density at each of its points, and $R^\infty(K, \mu)$ can be decomposed as a direct sum of an L^∞ term and an algebra isomorphic to $R^\infty(K, \lambda_{E(\mu)})$. In 1985 Cole and Gamelin gave a proof of Chaumat's theorem along the constructive lines of Sarason's proof, producing a chain of intermediate algebras which are invariant under the T_φ-operators used by Vitushkin to split singularities. (In a weak and uninspired moment someone dubbed these "T-invariant" algebras.) The theory so unified covers any algebra invariant under the localization operators.

Recently J. Thomson made a breakthrough on the L^2 polynomial approximation problem. He succeeded in answering an old question on the existence of analytic point evaluations, showing that if $P^2(\mu) \neq L^2(\mu)$ then there is a nonempty open set U on which the functions in $P^2(\mu)$ are analytic. This is a seminal result, which will have substantial spin-off. The basic idea of the proof was inspired by an earlier proof technique of M. S. Melnikov, who had shown in 1976 that each Gleason part for $R(K)$ is "area connected".

Now we turn to the book under review. It is an updated and expanded version of the Pitman notes, beginning with exactly the same dedication. It divides naturally into three parts.

The first four chapters (150 pages) constitute a rewritten version of the corresponding material in the Pitman notes. There is background material on function theory on the unit disk, general operator theory, and subnormal operators. There is also a chapter on hyponormal operators, for which the function theory has quite a different flavor.

The next two chapters (150 pages) comprise an exposition of uniform algebras and rational approximation theory with an eye toward developing topics of potential use in operator theory. Instead of focusing on polynomial approximation and avoiding analytic capacities as in the Pitman notes, the author bites the bullet and injects a strong dose of constructive approximation techniques à la Vitushkin. Applications to operator theory are woven into the discussion. Chapter V includes material on the localization operators and a complete proof of Davie's theorem; Mergelyan's theorem is also proved, though by the abstract route. Chapter VI includes the Chaumat theory for weak-star rational approximation and the Sarason theory for polynomial approximation.

The final two chapters (100 pages) are aimed at an analysis of subnormal operators with the aid of the machinery developed in the preceding chapters, following the work of the author, J. Dudziak, R. Olin, J. Thomson, and others. Chapter VII includes various spectral mapping theorems, a discussion of the minimal normal extension of $f(S)$, and the striking theorem of Olin and Thomson, dating to 1980, that subnormal operators are reflexive. The book culminates in an exposition of Thomson's theorem on the existence of analytic point evaluations, which occupies all of Chapter VIII.

I feel obliged as reviewer to serve up some critical comments, so here they are. Indeed I think that some reference should have been included to Melnikov's work, since it is so closely related to that of Thomson. Also, in Chapter VI both Chaumat's original proof and Sarason's constructive proof are given, whereas I (of course) would have preferred to have seen the unified approach mentioned above. Finally, if Otto Stolz were still around, he would undoubtedly be unhappy with the spelling of his name.

On the whole, the book is well written. The author has an engaging style, and the book is rife with folksy comments and pithy truths (e.g., "behind every beautiful theorem there is a grubby lemma"). Open problems are sprinkled here and there; one of the simplest, with obvious consequences for spectral mapping, is: If $f \in R^\infty(K, \mu)$ is bounded away from 0 on $E(\mu)$, does $1/f$ belong to $R^\infty(K, \mu)$?

In conclusion, the author has done us a service by preparing a digestible exposition of the current state of affairs in an interesting area at the confluence of function theory and operator theory. The book does not leave us with the impression that the theory is in a finished state, but rather that the area is an active and inviting research field with a lot of rough edges and surfaces begging to be smoothed out. Enjoy!

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Combinatorial matrix theory, by Richard A. Brualdi and Herbert J. Ryser. Cambridge Univ. Press, Cambridge, 1991, ix+367 pp. $49.50. ISBN 0-521-32265-0

This book explores the interrelations between the theories of graphs and matrices.

Graph theory is embedded in a not very well-defined region of mathematics called "combinatorics", which would seem to include the theory of sets of subsets of a given finite set. For example, many theorems about graphs can be stated in terms of circuits and edges only, the circuits being treated as subsets of the edge set satisfying some set-theoretical axioms. Generalized in a natural way, this variation becomes the theory of matroids. Matroids get only a passing mention in this book, but general families of subsets are important, for they have incidence matrices, defined and discussed in the first chapter.

In the early days of my acquaintance with matroids I supposed that by constructing their theory one superseded the theory of graphs and would no longer have to use it, so it was with a shock of surprise that I found myself forced to use what I considered a graph-theoretical argument in a proof of a theorem about matroids. I had a similar experience more recently in an attempt to clarify the Birkhoff-Lewis theory of "free" and "constrained" chromatic polynomials. Having transformed that theory into one about partitions of a cyclically ordered sequence, I claimed to have replaced all the graph theory in that problem by algebra. But then I needed to evaluate the determinant of a matrix defined by the relevant partitions. To my surprise I was unable to do this without going back to graph theory.

So I approached this book confident that it could not absorb graph theory into matrix theory. Conversely, I did not expect matrix theory to be shown as only an aspect of graph theory. To be sure though, I have heard it contended that all mathematics, properly presented, is graph theory; meaning, I suppose, that graph theory is a style of writing rather than a restricted region of mathematics.