not read like English at all, and many words are mistranslated, for example, simple for prime, conjunction for conjugation, boundary for bound, expanded for decomposed, directed for inductive, stempel for imprint, and many others. Definite and indefinite articles are constantly mixed up, and to add to all this there are numerous errors that careful proofreading should have caught; thus, the Queen's theorem turns out five pages later to be Quinn's theorem. The layout and type (word processor) also falls well short of a satisfactory standard. If properly typeset (or even done on a decent word processor), the book could have had a pleasing typeface and significantly less pages.

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An introduction to wavelets, by Charles K. Chui. Academic Press, New York 1992, x + 264 pp., \$49.95. ISBN 0-12-174584-8

Ten lectures on wavelets, by Ingrid Daubechies. CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, 1992, 357 pp., ISBN 0-89871-274-2

It would have been difficult to review the books written by Ch. Chui and L. Daubechies without describing the goals of the "wavelet venture". Wavelets were discovered in a signal processing context much before mathematicians became aware of their existence, and for this reason the first four sections of this review will be mostly science oriented. But an amazing discovery of the "wavelet revolution" has been the identification of the algorithms proposed by the signal processing community with some tools developed by the famous Calderón-Zygmund school. Sections 5–7 will be devoted to these relations. Chui's book will be analyzed in §8, Daubechies's in §9, and in §10 we give our personal feelings about the relevance of wavelets in science and technology.

1. WAVELET ANALYSIS VERSUS WINDOWED FOURIER ANALYSIS

Wavelet analysis can be defined as an alternative to the classical *windowed Fourier analysis*. In the latter case the goal is to measure the local *frequency* content of a signal, while in the wavelet case one is comparing several *magnifications* of this signal, with distinct *resolutions*. Wavelet analysis is related to *multiresolution analysis* in §6. The building blocks of a windowed Fourier analysis are sines and cosines (waves) multiplied by a *sliding window*. In a wavelet analysis the window is already oscillating and is called a *mother wavelet*. This mother wavelet $\psi(x)$ has a compact support (or a rapid decay at infinity), is smooth, and satisfies the fundamental condition $\int_{-\infty}^{\infty} \psi(x)\,dx = 0$, which means that in some weak sense $\psi(x)$ is oscillating. Moreover, the mother wavelet should also satisfy a technical but crucial condition, which will be revealed in §5. The mother wavelet is no longer multiplied by sines or cosines. Instead it is translated and dilated by arbitrary *translations* and *dilations.* That is the way the mother wavelet $\psi(x)$ generates the other wavelets $\psi_{a,b}(x) = a^{-1/2}\psi((x-b)/a)$, $a > 0$, $-\infty < b < +\infty$, which are the building blocks of a wavelet analysis. The parameter a measures the average width of the wavelet $\psi_{a,b}$, while the parameter b gives the position. These dilations (by $1/a$) are precisely the magnifications that we alluded to. The wavelet coefficients of a function $f(x)$ of the real variable x or of a signal $s(t)$ (t denoting the time variable) are the scalar products $W(a, b) = \langle f, \psi_{a,b}\rangle$, $a > 0$, $-\infty < b < +\infty$. The original function $f(x)$ can always be recovered as a linear combination of these wavelets $\psi_{a,b}(x)$ and, up to a normalization which will be specified in §5, the coefficients of this combination are precisely the wavelet coefficients $W(a, b)$.

A first objection against wavelet analysis is the fact that these coefficients $W(a, b)$ do not have any physical meaning. A second and related objection concerns their sensitivity to the choice of the analyzing wavelet. The Calderón-Zygmund theory will take care of this second objection.

A common wisdom among numerical analysts and image processing people is that *the inverse of a scale is a frequency*: small scales correspond to large frequencies and large scales to small frequencies. Moreover, very distinct scales should provide independent (i.e., nonredundant) information. Wavelet analysis could be defined as an attempt to give a very precise meaning to this folk belief.

2. Before wavelets

Many people are making big claims about the relevance of wavelet analysis in science and technology. These claims are at the same time correct and premature. Let me deepen this paradoxical contradiction:

(1) There are many examples of scientific discoveries or technological improvements that relied on wavelet-like techniques and were achieved a few decades or a few years before wavelets became popular.

(2) Today there are only a few interesting and original findings that depend on wavelets in their present form.

(3) There are some reasons for believing that wavelets will become a fundamental tool in science and technology.

The second statement can be easily checked by scanning the proceedings of any wavelet conference [6, 8, 15, 16]. The last remark will be developed below (§§9, 10). Returning to 1, indeed at least sixteen related concepts or algorithms have been known for decades, with other names. Before describing these sixteen topics, let us insist on the key words. In mathematics, wavelet-like expansions were introduced to face some *nonlinearities.* In quantum field theory, the problem of computing an L^4 norm of a fractional integral of a function of three real variables forced many teams to give up Fourier series expansions and to develop wavelet-like algorithms. In signal processing waveforms were introduced to store or compress signals with strong *transients.*

Let us provide the reader with some more details. In pure mathematics, three algorithms have been created to overcome some drawbacks of standard Fourier series expansions. These difficulties appear when one is facing the problem of measuring the size or the smoothness of a function. The simplest norms, based on quadratic estimates, can be extracted easily from Fourier coefficients. But as soon as L^p or L^∞ estimates are addressed, Fourier coefficients do not answer the problem and (1) the *Haar basis* (1909), (2) the *Franklin orthonormal system* (1927), or (3) the *Littlewood-Paley theory* (1930) are the correct tools.

Later on (4) *Calderón's reproducing identity* (1960) and (5) *atomic decompositions* (1972) were widely used in other functional settings (e.g., Hardy spaces). Both the Littlewood Paley theory and atomic decompositions are playing a key role in a branch of operator theory created by Calderón, Zygmund, and their school and named (6) *Calderón-Zygmund theory.* Just before wavelets became popular, J. O. Stromberg was using an almost identical recipe for solving a celebrated problem in (7) *geometry of Banach spaces* [the existence of a specific unconditional basis for the Hardy space $H^1(\mathbb{R})$].

In signal or image processing a similar and parallel evolution started from the standard windowed Fourier analysis and culminated into some discrete versions of Calderon's reproducing identity. Indeed D. Gabor introduced (8) *time frequency atoms in speech signal processing* (D. Gabor, 1946); Croisier, Esteban, and Galand developed (9) *subband coding* in signal processing (1975); and at about the same time Burt and Adelson described (10) *pyramidal algorithms* in image processing (1982). D. Marr was convinced that both the human vision and computer vision were based on similar algorithms, which should be in some sense independent from the "wires" to be used in their realizations. These specific algorithms are using the (11) *zero-crossings* of the wavelet transform of the 2-D signal (1982).

In numerical analysis, wavelets are related to (12) *spline approximation.* Before wavelets became fashionable V. Rokhlin created the so-called (13) *multipole algorithms. Refinement schemes* (14) play a key role in computer graphics.

Finally let us turn to mathematical physics. *Coherent states* (15) are fundamental in quantum mechanics. *Renormalization* (16) in quantum field theory is needed for extracting finite numbers from divergent integrals. It is based on some variants upon Littlewood-Paley techniques, which were mainly developed by K. Wilson, K. Gawedzki and A. Kupiainen, J. Glimm and A. Jaffe, and G. Battle and P. Federbush.

The interested reader may consult [13] and also [2, 3, 5, 7, 8, 15, 16] where one will find references for the aforementioned sixteen approaches to wavelets.

3. The wavelet revolution

The so-called "wavelet revolution" started with a beautiful collaboration between J. Morlet (an engineer working in oil prospecting for the Elf-Aquitaine company) and A. Grossmann (a leading physicist in quantum mechanics). Once mathematicians and signal processing experts entered the game, one of the main achievements of the "wavelet revolution" was to show that, up to some slight modifications, *the sixteen algorithms we listed above could be rephrased into the wavelet terminology and were telling the same story.*

But the price to be paid for reaching this "great synthesis" was to dilute the precise definition of the magic word "wavelet". For instance a "wavelet" in §5 will not be defined the same way as a "wavelet" in §6. Such a flexibility raises the fundamental issue of selecting the wavelet that gives the best fit to the problem to be solved by a wavelet analysis.

Once the "great synthesis" was accomplished, many people recognized some familiar ideas after listening to a lecture on wavelets and might claim that they were doing wavelets much before this new language was given its precise formulation. These claims are partly true and partly inaccurate. Let us give an example. Subband coding schemes and wavelet expansions are similar algorithms, but they are far from the being identical recipes with distinct names.

On the contrary very few quadrature mirror filters would cascade to smooth wavelets when the smallest scales tend to 0. Indeed most of the filters that were welcomed by signal processing experts are not leading to the smooth wavelets discovered by Ingrid Daubechies. The algebra was identical, but the scope was very distinct. M. Barlaud, A. Cohen, and Daubechies are investigating the performances of the wavelet approach to image processing, as opposed to the subband coding one [7]. Some other experts are claiming that both approaches would lead to similar performances, as far as compression is concerned.

Similarly a wavelet analysis is not an atomic decomposition, and some tricky regrouping is needed to get the one from the other.

4. Expectations from the wavelet revolution

People working on wavelets have successfully been able to modify sixteen algorithms that were commonly used in science or in technology in such a way that they would become sixteen chapters of a unified theory. In other words, an elegant and structured language originated from several dialects. The next step will be to prove that these modifications are relevant in science or technology. There is, however, a difficulty for accepting this belief since the *wavelet synthesis relied upon tiny changes on existing methods and not on dramatically new ideas.* Do such tiny changes improve the existing algorithms? Some people say that the only impact of the wavelet revolution has been the clarification of some methods and the gain in perspective. As stated above, in spite of the fact that subband coding and wavelets do not tell exactly the same story, some experts think that both approaches to image processing would yield similar performances. An excellent discussion of what wavelets might bring to image processing, when compared to more conventional methods, was presented by Unser and Aldroubi in [5, 8].

Now let me forget these pessimistic remarks and indicate two reasons for believing that science might benefit from the wavelet approach [in §10 this issue will be addressed again].

• First wavelets are without any doubt an exciting and intuitive concept. *This concept brings with it a new way of thinking*, which is absolutely essential and was entirely missing in the previously existing algorithms. For that reason, wavelets will be more appealing than other algorithms to engineers who are still using a conventional FFT or JPEG in a routine way and do not even know subband coding. These people could have accepted Littlewood-Paley theory instead, but the latter was confined inside pure mathematics and the signal processing community was unaware of its existence.

An example of a true scientific discovery obtained by wavelet methods was made in astronomy by Slezak and his collaborators [1]. It is a little bit frustrating for a mathematician to observe that in this case, the wavelet analysis that was used was by far the most primitive. But this primitive analysis was adapted to the problem and worked much better than any one of the existing tools in image processing.

• A second positive aspect of the wavelet revolution is its dynamism. Once a revolution is launched, it often goes far beyond its first goals; and that observation can be applied to the wavelet venture. As will be stressed now, we have already gone beyond wavelets.

For instance the obligation to face nonacademic problems forced the wavelet experts to use "conventional wavelets" in a nonstandard way or to give up "conventional wavelets". Conventional wavelets are "time-scale atoms" and they are adapted to situations where some relevant information is obtained by "zooming in" a given signal or image at distinct scales. In other contexts, for instance in texture discrimination, a wavelet approach is inappropriate and the *time-scale atoms* should be replaced by *time-frequency atoms.*

These considerations led the wavelet pioneers to develop efficient time frequency algorithms. The Marseille team around Grossmann discovered a wavelet-based method for extracting the modulation laws for large classes of signals. This method cannot be developed within a subband coding way of thinking [8].

Another interesting example is the fingerprints contest raised by the FBI. The problem was to compress and to store digitalized fingerprints. All existing algorithms (JPEG, subband coding, etc.) were tested by image processing experts. The winning algorithm was designed by the Yale team, around R. Coifman. The algorithm is based upon "wavelet packets", a smooth version of the Walsh orthonormal basis. Another option is given by the *local cosine basis.* These completely new methods are described in Daubechies's book [7].

The *best basis selection algorithm* [8] is the second ingredient of the solution, and it uses the full library of all wavelet-packets bases. It means that the standard wavelets were not efficient for this specific problem and that a search meta-algorithm was needed to select the best algorithm [8].

In image compression, the role played by standard wavelets is not clear, since the large compression gains obtained by M. Barlaud and his team [7] rely on vector quantization and it is not that obvious if wavelets have added something essential to subband coding.

5. Continuous wavelet analysis

A wavelet analysis is either continuous, semidiscrete, orthonormal, or biorthogonal. In the first case one uses *Calderón's reproducing identity.* For the reader's convenience, let us stress the relationship between this identity and a wavelet analysis and take this opportunity for giving the precise definition of a wavelet. An *analyzing wavelet* $\psi(x)$ is a function in $L^2(\mathbb{R}^n)$ whose Fourier transform $\hat{\psi}(\xi)$ satisfies $\int_0^\infty |\hat{\psi}(t\xi)|^2\, dt/t = 1$ for every $\xi \neq 0$. Then the continuous wavelet coefficients of a function $f(x)$ in $L^2(\mathbb{R}^n)$ are defined as $F(x, t) = \langle f, \psi_{x,t}\rangle$ where $\psi_{x,t}(y) = t^{-n/2}\psi((y-x)/t)$. For recovering $f(x)$ it suffices to combine all the wavelets $\psi_{x,t}$ with precisely these coefficients. In other words $f(y) = \int_0^\infty \int_{\mathbb{R}^n} F(x, t)\psi_{x,t}(y)\, dx\, dt/t^{n+1}$, which is exactly Calderón's reproducing identity. The relevance of a continuous wavelet analysis will depend heavily on the properties of the analyzing wavelet $\psi(x)$. Two usual choices are the Morlet wavelet (a modulated gaussian) or the Mexican hat (the second derivative of a gaussian).

What Calderón's reproducing identity tells us is the following: *a wavelet analysis gives a recipe for measuring the local fluctuation coefficients of a given function* f, *around any point* x, *at any scale* t *and for reconstructing* f *with* *all these fluctuation coefficients.* In other words at any given scale $a > 0$, f is decomposed into the sum of a *trend* at the scale t and of a *fluctuation* around this trend. The trend is given by the contribution of scales $t > a$ in Calderón's reproducing identity, and the fluctuation is given by the scales $t < a$.

6. Multiresolution analysis

Let us jump over the discrete versions of Calderón's identity, which receive a superb description in Daubechies's book [7], and switch to *orthonormal wavelets,* as an example of a remarkable mathematical discovery obtained by the "wavelet revolution".

In contrast to what has been stated in §5, a "mother wavelet" will now be defined as a function $\psi(x)$ enjoying the three following properties:

(1) $\psi(x)$ is a *smooth* function (with $r-1$ continuous derivatives and a bounded derivative of order r);

(2) $\psi(x)$ together with its derivatives of order $\leq r$ has a *rapid decay at infinity*;

(3) the collection $\psi_{j,k}$ defined by $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, $j = 0, \pm 1, \pm 2, \pm 3, \dots$, $k = 0, \pm 1, \pm 2, \pm 3, \dots$ is an *orthonormal basis for* $L^2(\mathbb{R})$.

The first problem in the theory is to construct such functions $\psi(x)$, and the second one is to show that the wavelet coefficients yield relevant information. The very first example of a mother wavelet was given by A. Haar in 1909. The Haar wavelet $h(x)$ is defined by $h(x) = 1$ on $[0, \frac{1}{2})$, $h(x) = -1$ on $[\frac{1}{2}, 1)$ and $h(x) = 0$ elsewhere. In that case $r = 0$. But about eighty years were needed until Daubechies [7] proved that, for each $r \geq 1$, one can construct a function $\psi(x)$ of class C^r with compact support and satisfying the above conditions (1) and (3), the second condition being obvious. A detour with a visit to the signal processing community and a reshaping of the subband coding algorithms were needed to build these Daubechies wavelets (the best reference is Chapters 5–8).

Today we know that this detour is absolutely necessary. Lemarie [11] proved that if $\psi(x)$ satisfies (1), (2), and (3) and is compactly supported, there exists a *multiresolution analysis* behind this orthonormal basis.

A multiresolution analysis is a ladder V_j, $j \in \mathbb{Z}$, of closed subspaces of $L^2(\mathbb{R})$ enjoying the following four properties:

(4) the intersection of all the V_j's is reduced to $\{0\}$;

(5) the union of these V_j's is dense in L^2;

(6) $f(x)$ belongs to V_j if and only if $f(2x)$ belongs to V_{j+1}; and

(7) there should exist a smooth and localized function $\varphi(x)$ such that the collection $\varphi(x-k)$, $k \in \mathbb{Z}$, be an orthonormal basis for V_0.

Multiresolution analysis is a natural concept for people working on splines since refinements of meshes provide trivial examples. The relation between our wavelet basis and a multiresolution analysis is given by the condition that $\psi(x-k)$, $k \in \mathbb{Z}$, is an orthonormal basis of the orthogonal complement W_0 of V_0 in V_1. By an obvious rescaling one obtains the fact that $2^{j/2}\psi(2^j x - k)$, $k \in \mathbb{Z}$, is an orthonormal basis for the orthonormal complement W_j of V_j into V_{j+1}. It is then clear that the full collection $\psi_{j,k}$ is an orthonormal basis for L^2.

The quadrature mirror filters used in subband coding give the matrix representation of the orthonormal decomposition $V_{j+1} = V_j \oplus W_j$. The multiresolution analysis used by Daubechies to construct her wavelets is highly nonstandard and was never considered by "spline people" who where unaware of the work achieved by signal processing researchers on quadrature mirror filters. Moreover, spline specialists were not interested in the spaces W_j, giving the missing details needed for a "coarse to fine algorithm".

This connection between wavelet analysis, multiresolution analysis, and subband coding was stressed by Mallat, and the filters that produce $\psi(x)$ by a cascade algorithm were not unfamiliar to the signal processing community. The reason why they did not discover Daubechies wavelets seems due to the fact that they did not know that any signal could be decomposed into a sum of wiggling waveforms which would be obtained by dilations and translations from a mother wavelet $\psi(x)$. This idea, however, was quite familiar to mathematicians around Guido Weiss who had been studying atomic decompositions for about ten years. But most of the atoms used by Guido Weiss do not satisfy the smoothness conditions that we imposed on wavelets. This example shows that in spite of the fact that the filters that are needed for building Daubechies wavelets were implicit in existing algorithms, the "wavelet wisdom" was essential to reveal these remarkable orthonormal bases.

7. Wavelet analysis and Calderón-Zygmund operators

One cannot expect any serious understanding of what wavelet analysis means without a deep knowledge of the corresponding operation theory. As it is stressed in Daubechies's book, writing a partial sum of a wavelet expansion means doing a specific *time-frequency localization.* Time frequency localization operators should definitively be understood before deriving any conclusion from a wavelet analysis. In the wavelet case, these operators happen to be Calderón-Zygmund operators, while in the windowed Fourier transform case, the corresponding operator theory is much wilder [it corresponds to the symbols $S^0_{0,0}$ in Hormander's terminology]. A second remark explains the crucial role played by these Calderón-Zygmund operators. Sticking to the wavelet case, let us observe that there are several interesting choices of orthonormal wavelet bases and that one needs to know if some results obtained by using one specific basis would still be true with another one. For answering this problem one needs a dictionary between all those bases. This dictionary is provided by the *Calderón-Zygmund theory.* One key ingredient in this operator theory is the ability of *rescaling* global L^2-estimates for obtaining pointwise information. Therefore, as stressed by E. Stein, the group of dilations plays a crucial role in the Calderón-Zygmund theory. The Hilbert transform H, defined by $\pi H(f)(x) = \text{p.v.}\int_{-\infty}^{\infty} f(x-y)\, dy/y$, is the prototype of a Calderón-Zygmund operator. It is the only nontrivial operator that is translation and dilation invariant (only positive dilations are considered). If $\psi(x)$ is a mother wavelet generating an orthonormal wavelet basis, its Hilbert transform $H(\psi) = \tilde{\psi}$ is another one and H maps the orthonormal collection $\psi_{j,k}$ onto $\tilde{\psi}_{j,k}$. Therefore, all the information obtained by an orthonormal wavelet analysis is necessarily invariant under the action of the Hilbert transform. *For example, one cannot decide if a function is continuous by inspecting its wavelet coefficients.* Indeed continuity is not preserved by the Hilbert transformation. Similarly it is impossible to obtain wavelet coefficient-based criterion for deciding if a given distribution is a measure. On the other hand, Hölder classes are preserved by the Hilbert transformation, which means that Hölder exponents can be computed through inspecting a wavelet expansion. This explains why wavelets are playing a key role in fractal analysis where local Hölder exponents are computed. The Hilbert transformation belongs to the group Γ of those unitary Calderón-Zygmund operators which are mapping any orthonormal wavelet basis into any other one. The interested reader is referred to [14]; *it is quite unfortunate that the fundamental issue of the invariance of any wavelet analysis with respect to the group Γ was never discussed by wavelets experts.*

A final remark concerns quantization. It means that whenever a computer is used, the true coefficients arising in some expansion will be replaced by approximations to a given precision. What happens to the expansion after a quantization is performed is a problem to be addressed. Some representations are more sensitive than others to quantization. Wavelet expansions have the advantage that the effect of small changes over the coefficients will only have a local influence. When the trigonometric system is used, any change on any coefficient will everywhere affect the resulting function. These remarks also depend on the Calderón-Zygmund theory. A deep understanding of the relations between data compression and wavelet shrinkage has been provided by Donoho and Johnstone [9]. Another reference is De Vore, Jawerth, and Popov [8].

8. Chui's book

Chui is one of the leading experts on splines, and his book stresses the relations between the construction of wavelets and traditional splines. In other words Chui is concentrating on *two* [(12) and (14) in the previous list] out of the sixteen roads leading to wavelets. Chui's thesis is original. He claims that completely explicit biorthogonal spline wavelets should be preferred to the orthogonal Daubechies wavelets. Daubechies wavelets are fascinating but do not have any simple analytical form. Chui is strongly advocating for some compactly supported semiorthogonal spline wavelets, and his Chapter 6 is so convincing that I have been tempted to give up Daubechies orthogonal wavelets. Moreover, the "Chui wavelets" are extremely close to the (real or imaginary part of) Morlet's wavelets ([5] or [8]). Let me give one example of these nice Chui wavelets. In this example, the mother wavelet $\psi(x)$ is continuous, supported by $[-1, 2]$; it satisfies $\psi(1-x) = \psi(x)$ (the so-called linear phase condition); it is affine on each interval $[k/2, (k+1)/2]$, $k \in \mathbb{Z}$; and the values at -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2 are respectively $0, 1, -6, 10, -6, 1, 0$. In that specific case, the sequence $\psi(x-k)$, $k \in \mathbb{Z}$, is a Riesz basis for a closed subspace W_0 of $L^2(\mathbb{R})$, and this subspace is obtained by the procedure described in section 6, applied to the multiresolution ladder based on linear splines. The only drawback with Chui's wavelets is the fact that the biorthonormal system does not have compact support. In other words the filters, which are needed for the subband coding schemes, do not have finite length. Therefore, I would hesitate between Chui's wavelets (which were previously known by Lemarié, ...) and other biorthogonal wavelets, which do not show up in Chui's book. These biorthogonal wavelets are advocated by Cohen, Daubechies, and Feauveau [7] and implicitly contained in a celebrated paper by Burt and Adelson [2]. They have the following nice properties: all filters have finite length, the wavelet itself

$\psi(x)$ is a spline function with compact support, and the dual wavelet $\tilde{\psi}(x)$ has a compact support; however, the dual wavelet has a fractal type structure and is no longer a spline function.

In Chui's wavelet case, the distinct dyadic channels W_j generated by the mother wavelet are orthogonal. In the Cohen-Daubechies-Feauveau case the corresponding channels are almost orthogonal. It seems impossible to decide what the best choice is. In my opinion, this decision should be left to the scientist who is using a wavelet analysis.

Chui's book is a list of recipes for cooking some wavelets. What Chui does not tell you is why you are building these wavelets. In other words, Chui takes for granted that wavelet analysis is already an important tool in science and engineering. This importance is far from being proved; even if I would bet on wavelets on the long range, I have to stress that big claims are numerous, while there are few examples of technological or scientific advances obtained through wavelets. This criticism against Chui's book should not be taken too seriously though since Chui's book is followed by a second one, of which Chui is the editor, entitled *Wavelets, a tutorial in theory and applications*, that contains some interesting applications [5].

9. Daubechies's book

We now arrive at Daubechies's book. It is written by a physicist who moved to mathematics. Indeed $\int dx f(x)$ instead of $\int f(x)\,dx$ is not a misprint! Daubechies is also a member of the signal processing community. There are deep connections between quantum mechanics and signal analysis. Let me give two examples of these relations:

(1) The limitations given by the Heisenberg uncertainty principle apply to both fields. In other words, the impossibility of measuring simultaneously the position and the momentum in quantum mechanics is reflected by the impossibility of defining the instantaneous frequency in signal processing.

(2) The Wigner-Ville analysis of signals is an adaptation of Weyl's quantification.

These remarks imply that an expertise in quantum mechanics is absolutely necessary to get a better understanding of signal analysis. Both Daubechies and Grossmann, who were leading experts in quantum mechanics, successfully switched to signal processing. Among the peak points of Daubechies's book is the proof by G. Battle (also a specialist of quantum field theory) of the Balian-Low theorem. This theorem states that orthonormal bases cannot be extracted from a windowed Fourier analysis. Daubechies has a profound understanding of almost all the papers written on wavelets. In contrast with Chui, who is advocating two roads for entering the wavelet world and one specific choice for the analyzing wavelet, Daubechies is offering the reader most of the sixteen approaches to wavelets that are listed above. As mentioned before, a special emphasis is given in her book to the relations with the theory of coherent states in physics but also to specific and concrete problems arising in signal processing. For instance, her description of *time-frequency localization algorithms* is excellent. More than half of the book is devoted to the construction of orthonormal wavelet bases, with prescribed regularity and compact support; but other options are provided, like *biorthogonal wavelets, wavelet packet bases, localized sine bases*, etc.

The interdisciplinary spirit of the wavelet community is beautifully alive in her presentation. Her book is written with a true care for the reader, and her style is extremely warm and original, with a remarkable emphasis on the heuristics. The book will obviously be appealing to engineers, physicists, and mathematicians and will be welcomed as being "the book" on wavelets.

If I would be allowed to make a few criticisms, Daubechies does not tell you in what specific context one should prefer orthonormal wavelets to frames or to continuous wavelets. The reason is about the same as in Chui's book. Her book was intended to be a companion to the proceedings of the Lowell conference on wavelets and their applications [16]. The reader interested in those applications is advised to read [6, 8, 12, 13, 15, 16]. Nevertheless, he will not find the answer to the problem I raised, and it seems that nobody knows, even in the case of image processing, which has been the most studied. Do specific tasks require specific wavelets?

10. Wavelets, science, and technology

Let me return to the relevance of wavelet analysis in science and technology. As was already clear in §2, wavelet analysis was used much before it was given its name. A lot of scientific work where wavelet-like methods are playing a key role was completed before the wavelet phraseology became popular, and we are eager to see if translating this work into the wavelet language will bring some improvements. Some people are saying that the wavelet spirit was already there, while other scientists feel that some improvements could be obtained from the fact that the wavelet way of thinking offers a lot of freedom for reshaping the previously existing algorithms.

So much for the past and we may turn to the future. There are two reasons for being much more optimistic. The first one has been indicated (§4): the dynamism brought by the wavelet revolution will continue, and other important tools will show up far beyond wavelets. The second one is the observation that wavelets are popular among young people. Young scientists have no problem using a wavelet analysis while they would have been more reluctant to extract similar tools inside some deep paper written by A. Calderón. Peter Jones defined the merit of wavelet analysis by the quality of being trivial: anyone can do it; it is like driving a car. As was clear during the 1992 Toulouse conference, wavelet analysis is no longer a chapter of science but should be better viewed as a tool or a trick, like integration by parts. It will be used again and again and will play the modest but fundamental role of the furniture in a house.

Young scientists need an easy access to valuable information on wavelets. *The merit of Chui's book and of Daubechies's monograph is to provide these young scientists with exactly what they are looking for.*

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Scenes from the history of real functions, by Fyodor A. Medvedev. Birkhäuser Verlag, Basel, 1991, 265 pp., $98.00. ISBN 3-7643-2572-0

The book is a translation by Roger Cooke from the Russian original published in Moscow in 1975 with the title *Essays on the history of the theory of functions of a real variable.*

The central concern of the book is with real functions of a real variable, but in places it leads into some aspects of functional analysis. For example, consideration is given to the function spaces L^2 and L^p and to work of W. H. Young on the notion of conjugate pairs of classes of functions such that the product of two functions f, g, one from each of the two classes of a pair, is integrable in the sense of Lebesgue. Medvedev points out that in a work by Burkill, developing the ideas of Young, there was an error, the correction of which by Birnbaum and Orlicz accompanied the creation of Orlicz spaces.

In the first chapter of his book Medvedev regards the theory of functions as a subject to be distinguished, although not very precisely, from what he refers to as classical analysis. He asserts that if from classical analysis is excluded