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An introduction to multigrid methods, by P. Wesseling. Wiley, New York, 1992, vii+284 pp., $89.95. ISBN 0-471-93083-0

The term *multigrid* refers to a numerical technique that uses a family of grids of differing mesh sizes to discretize and solve continuum problems, most notably partial differential equations. The basic idea is to use a few iterations of an inexpensive *relaxation* scheme (e.g., Gauss-Seidel) on each grid level to attenuate error components of the approximation that vary on a scale comparable to the associated mesh size. These relaxations are performed for equations defined on each grid that approximate a suitable finest-grid error equation (e.g., the residual equation for linear problems), and each result is interpolated to the finest grid to correct the current approximation there. The attraction of these methods is that they are often optimal in the sense that they can produce a result whose accuracy is comparable to the finest-grid discretization error at a cost equivalent to a few finest-grid relaxations. This optimality can often be obtained over a wide range of problem difficulties, including nonselfadjoint operators, nonlinearities, various discontinuities and singularities, and local refinement.

The first known multigrid scheme was developed by Southwell [15] in 1935 for equations of elasticity. Understandably, it was primitive by current standards in its rather awkward use of just two discretization levels. Multigrid schemes using more than two levels were apparently first devised in the 1960s for Poisson's equation by Fedorenko [6] and for more general elliptic equations by Bakhvalov [1], although these methods were still too ineffective to attract much attention. The modern era of truly efficient multigrid techniques was pioneered by Brandt in the 1970s [2, 3]. Since then, multigrid has become the method of choice for a wide range of problems; and its generalizations, referred to by such terms as *multilevel, multiscale*, and *multiresolution*, have begun to invade many diverse disciplines, including aerodynamics, chemistry, civil engineering, economics, geology, image processing, and statistical physics.

Expository publications have lagged well behind technological progress in the multigrid discipline. For several years, the seminal paper by Brandt [3] in 1977 served as the main resource for obtaining an understanding of the practical aspects of multigrid methodology. It contained many of the basic tools and principles that now constitute the core of the discipline, although most of the topics were naturally treated in brief. The "Yellow Book" [8] appeared in 1982 and quickly became the most popular general resource of that decade.

Among other important contributions, it included a more in-depth and very useful treatment of multigrid fundamentals by Stüben and Trottenberg, a general development of state-of-the-art theory by Hackbusch, and a practical guide to more sophisticated algorithms and principles by Brandt. The chapter by Brandt was expanded in 1984 into a much more comprehensive "Guide" [4] that emphasized computational fluid dynamics. Hackbusch's book [7], which appeared in 1985, focused on a sophisticated mathematical development of multigrid methods together with several important applications.

These and other materials each played a pedagogical role at some level, although none provided an easy-to-read introduction to multigrid concepts for a wide audience. It was not until the 1987 publication of Brigg's "Tutorial" [5] that a fundamental understanding of multigrid and its basic principles became accessible to the general scientific community. Yet there remained a gap between this introductory-level tutorial and advanced-level material like Hackbusch's book and Brandt's guide.

Wesseling's new book is an attempt to fill this gap. In the author's beginning words:

> The purpose of this book is to present, at graduate level, an introduction to the application of multigrid methods to elliptic and hyperbolic partial differential equations for engineers, physicists and applied mathematicians. The reader is assumed to be familiar with the basics of the analysis of partial differential equations and of numerical mathematics, but the use of more advanced mathematical tools, such as functional analysis, is avoided. The book is intended to be accessible to a wide audience of users of computational methods. We do not, therefore, delve deeply into the mathematical foundations.

The focus of Wesseling's book is on elliptic and hyperbolic partial differential equations discretized by finite difference and finite volume methods. In concrete terms, the book describes all basic multigrid components, including discretization of the equations and boundary conditions, relaxation schemes, intergrid transfer operators, and coarse-grid equations. It also provides a detailed exposition of the analytical tools used by the multigrid community, including abstract theory, realistic Fourier analysis, and numerical observation, with lots of illustrative examples. Special emphasis of this book is on the Fourier smoothing analysis of a wide variety of relaxation methods, with the aim of providing a fairly complete resource to guide the choice of an effective smoother for a given application. Also included are descriptions of multigrid schemes for various problems of fluid flow, including full potential, Euler, Boussinesq, and incompressible and compressible Navier-Stokes equations.

Wesseling's book should be read by anyone interested in exploring the multigrid discipline beyond the scope of a basic introduction. As such, it provides an important foundation in practical understanding and ability to tackle conventional applications, as well as a bridge to deeper aspects of the discipline provided by the more advanced materials now available. (In addition to those mentioned above, reading beyond Wesseling's book might include the 1987 SIAM text [12] that contains an extensive historical bibliography, two SIAM monographs [13, 14], and the proceedings of the latest European [9] and Copper Mountain [10, 11] conferences.)

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Equivariant surgery theories and their periodicity properties, by K. H. Dovermann and Reinhard Schultz. Lecture Notes in Mathematics, vol. 1443, Springer-Verlag, New York, 1990, 225 pp. $24.00. ISBN 3-540-53042-8

The book under review, *Equivariant surgery theories and their periodicity properties*, by K. H. Dovermann and Reinhard Schultz is a contribution to surgery theory.