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STEVE MCCORMICK
UNIVERSITY OF COLORADO AT DENVER
*E-mail address*: smccormi@copper.denver.colorado.edu

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 2, April 1993

0273-0979/93 $1.00 + $.25 per page

*Equivariant surgery theories and their periodicity properties*, by K. H. Dovermann and Reinhard Schultz. Lecture Notes in Mathematics, vol. 1443, Springer-Verlag, New York, 1990, 225 pp. $24.00. ISBN 3-540-53042-8

The book under review, *Equivariant surgery theories and their periodicity properties*, by K. H. Dovermann and Reinhard Schultz is a contribution to surgery theory.

The subject of surgery itself has a fascinating history and has been central in the flowering of high-dimensional geometric topology over the past two and a half decades. Surgery is essentially an elaborate calculus of manifold cutting and pasting which allows one to reduce geometric questions to algebraic ones, often of a very difficult and complex nature. The underlying concepts and the relationship to fundamental questions in geometric topology, however, are quite simple and direct. This permits a nontechnical exposition from a historical point of view.

I think an informal sketch of this background and considering the book within this general context, in an attempt to explain its concerns to a broader mathematical public, will be more useful, and certainly more fun, then a critique directed only toward the experts. The specialists, in any case, will want to peruse the book individually and debate the fine points, as civility demands, out of public view. Let me say in advance that the following is in no way intended to be more than a very rough sketch of a big and often complicated picture. I apologize in advance to those whose significant contributions in this area would be noted in a well-rounded treatment but are not acknowledged here.

High-dimensional geometric topology (HDGT) is the study of the topological classification and symmetries of manifolds, of dimension five and larger. The development of HDGT can be conceptualized into four distinct periods. The first period, which ran from the late 1950s through the mid-1960s, was focused and highlighted by the success of Smale, Stallings, and others (see Mi2 for details) in settling the Poincaré conjecture in dimension five and larger; that is, they showed that the sphere is topologically characterized among closed manifolds in terms of its homotopy type, which in the case of the sphere is determined by the vanishing of its fundamental group and most of its homology. This work led to further very striking and remarkable results and, in particular, to the discovery of the powerful technique of surgery which was to play such a crucial role in subsequent development.

This came about as follows. Milnor, using ideas of Thom, discovered the anomaly that while, according to Smale, the homotopy type of $S^n$ characterized it up to *homeomorphism* among closed manifolds, in dimension seven and larger it carried a number of essentially distinct differentiable structures. On the other hand, the fundamental result of Smale, the $h$-cobordism theorem, showed that the $n$ disk, $D^n$, is characterized smoothly by its homotopy type, i.e., by the fact that it is contractible to a point and has simply connected boundary.

Milnor, in collaboration with Kervaire, had the idea that one could use this smooth characterization of $S^n$, namely, that it bounds a contractible $(n+1)$-dimensional manifold, to describe the set (actually it turns out to be an Abelian group) of smooth $n$ manifolds homotopy equivalent and thus homeomorphic to $S^n$. By homotopy theory they could show that this group of homotopy $n$ spheres had a subgroup of finite index consisting of those homotopy $n$ spheres which bound a parallelizable manifold. If $\Sigma$ bounds such a $W$, is it possible to do something to $W$ without affecting $\partial W = \Sigma$, to make it contractible?

If $W$ is not contractible, then it contains an essential spherical mapping $S^k \to W$. If $k < \frac{n+1}{2}$, this sphere can be taken as an embedded sphere. Since $W$ is parallelizable, $S^k$ has a normal bundle in $W$ of the form $S^k \times R^{n-k+1}$. If one removes the interior of $S^k \times D^{n-k+1}$ and replaces it by $D^{k+1} \times S^{n-k}$, which has the same boundary as $S^k \times D^{n-k+1}$, one gets a new manifold $W'$ still having $\Sigma$ as its boundary. This procedure is called *surgery*. If $S^k$ is a smallest-dimensional essential sphere in $W$, then this procedure actually kills it without adding any new essential spheres in dimensions $< k+1$, provided $k < [\frac{n+1}{2}]$. Thus this procedure can be continued until you get up to half the dimension of $n$ or $n+1$. In these dimensions you run into obstructions which Milnor and Kervaire are able to calculate, at least for $n > 3$. These obstructions lie in Abelian groups, $L_{n+1}$, which depend only on $n+1$ modulo $4$. In fact $L_0 = Z$, $L_2 = Z/2$, $L_1 = L_3 = 0$.

What happens for $k >$ half of $n$, $n+1$? Here Poincaré duality, or more precisely its version for manifolds with boundary, Lefschetz duality, kicks in to tell you there are no essential spheres in these dimensions once you have killed the essential spheres in lower dimensions.

It was quickly realized by Browder, and independently by Novikov, that this method of surgery could be applied far more generally. In particular, suppose one has a map of compact manifolds with boundary $f\colon (W, \partial W) \to (V, \partial V)$ such that $f|\partial W \to \partial V$ is a homotopy equivalence. The $V$ corresponds to $D^{n+1}$ in the Kervaire-Milnor situation. (We hid the map $f$ there. Since $D^{n+1}$ is contractible, it did not really enter in.) We also need an analogue of the parallelizability condition; in particular, the tangent bundle of $W$ pulls back via $f$ from some bundle over $V$, possibly but not necessarily the tangent bundle of $V$. Under these circumstances one wants to measure the obstruction to changing $W$ and $f$, without changing the boundary, so that $f$ becomes a homotopy equivalence. This is called a *surgery problem.* The fundamental fact is that the obstructions lie in the same groups discovered by Milnor-Kervaire, $L_{\dim V}$, *provided* $V$ *is simply connected, i.e., provided* $\pi_1(V) = 0$.

The most important case of this is when $V = M \times I$. By the $h$-cobordism theorem of Smale (which resolved the Poincaré Conjecture mentioned above) and homotopy theory, these surgery obstructions classify manifolds of the same homotopy type of $M$ up to an indeterminacy depending only on the homotopy type of $M$, at least when dimension $M > 4$ and $\pi_1(M) = 0$. More precisely, the classification of such manifolds is reduced to Algebra, i.e., the $L$ groups, and the data about the tangent bundles, the tangential indeterminacy, which is controlled by the homotopy type of $M$.

Two gaping holes remained in the project for classifying higher-dimensional manifolds. This became the focus of the second period of HDGT. The first was to generalize to the case of nontrivial fundamental groups. This task, which was the most technically difficult part of the project, was carried out by Wall, who constructed surgery obstruction groups $L_n(\pi)$ depending on the fundamental group $\pi = \pi_1(M)$. The calculation of these groups became a major technical task and achievement of this period. The second was to clarify the nature of the tangential indeterminacy. This was done by Sullivan, and here a most striking phenomenon appeared. It was known from the work of Kervaire-Milnor that this indeterminacy involved the higher homotopy groups of spheres, which is unfortunate, since these groups are complicated and not easily calculable. What Sullivan discovered was that if you wish only to classify manifolds up to piecewise linear homeomorphism, a weaker notion than diffeomorphism, this indeterminacy was completely calculable, and Sullivan made this calculation in terms of classical invariants, more precisely in terms of certain cohomology classes associated naturally to the manifold and certain "characteristic" submanifolds.

The haunting question then remained: Whether one could replace the piecewise linear category, which was employed because the surgery techniques worked there, with the more natural and fundamental topological category. This was resolved affirmatively, through the work of Novikov, Sullivan, Lashoff-Rothenberg, and Kirby-Siebenmann, who obtained the definitive results. These results depended crucially on the fact that under rather general conditions *a manifold that is homotopic to the product of a closed simply connected manifold and a torus actually is such a product*, and this in turn relied on surgery theory and in particular on the calculation of $L_n(Z^m)$. Thus once again surgery thrust itself into the center of HDGT. The successful resolution of these problems brought to an end the second period of HDGT.

The third period of HDGT actually grew out of the work of Wall, who realized that surgery could be a powerful tool in the study of group actions on manifolds, the fundamental group acting on the universal cover being a central example of such phenomena. His classification of finite cyclic groups acting freely on spheres was a tour de force which established surgery theory as central in transformation groups. This line of work culminated in the results of Madsen-Thomas-Wall, which resolved the spherical space form problem, i.e., gave a complete classification of which finite groups could act freely on the sphere.

These new methods and results drew geometric topologists, and in particular many of the most talented of the younger ones, into transformation groups, throughout the 1970s. Reinhard Schultz, a student of Lashoff and the senior of the authors of the book under review (henceforth denoted DS), was among them. He was to make major contributions throughout this period.

A central technical problem that remained to be confronted was that the existing surgery methods were modeled on the action of the fundamental group on the universal cover. Therefore they directly applied only to free actions, i.e., those actions for which no point is left fixed by any group element other than the identity. From the point of view of transformation group theory, this is an artificial and unpleasant restriction. The project of constructing a surgery theory that applies to the more general situation, an equivariant surgery theory, thus became an important concern. This is specifically the project to which DS is a contribution.

The first serious attempt to construct such a surgery theory was that of Browder-Quinn back in 1973. Their theory has remained influential and useful because it shares the basic, simple properties of the nonequivariant theory, although it suffers from some rather damaging drawbacks. Most important, the category of allowable maps is severely restricted. A more ambitious project was initiated by Petrie in the mid-1970s and continued into the early 1980s in collaboration with Dovermann, then a young colleague at Rutgers. Their theory allowed a broader and more useful category of maps but put restrictions on the manifolds. Precisely they needed large gaps in the dimensions of fixed point sets (i.e., the gap hypothesis). Both these theories were also essentially theories for finite group actions, a limitation that has become more and more telling in recent years.

The Browder-Quinn theory, after a number of years of lying dormant, has been substantially rehabilitated and strengthened recently by Quinn and then Weinberger to remove many of the more onerous restrictions. It remains, however, basically an isovariant theory, that is, the maps are required to preserve orbit types. The Petrie theory, after a period of development, has been recently cast in an elegant and relatively finished form by Luck-Madsen. The gap hypothesis remains necessary for the general theory, although in the category of isovariant maps it is not required. Conversely Browder, and in a different form Dovermann, have shown that in the presence of gaps, the isovariant restriction is not much of a restriction. Thus, in essence, both theories suffer from the same limitation.

Despite the lack of finished theories, workers in the area of finite transformation groups on manifolds continued to get exciting and interesting (even sensational) results using surgery methods, often ad hoc, into the late 1980s. Perhaps the most dramatic were the examples of Cappell-Shaneson of distinct linear representations of $Z/4k$, $k > 1$, which are topologically equivalent. However, by the end of the 1980s the application of surgery methods for the study of finite group actions appeared to have become exhausted, and the third period of HDGT came to an end.

We now come to the current epoch. During the mid- and late 1980s the most exciting new work in HDGT became focused on noncompact manifolds and, in particular, infinite discrete group actions. Various topological versions of Mostow rigidity were explored concerning nice actions of such groups on Euclidean space, these versions going under the title of Borel conjecture or Novikov conjecture. The Borel conjecture asserts that there exists at most one, up to conjugacy, action of a given group on $R^n$, with compact quotient space, which is free and preserves some Riemannian metric. The Novikov conjecture is a stabilized version of the Borel conjecture which implies the same conclusion after multiplying by $R^3$. Pioneering work has been done earlier in the U.S. by Hsiang and Farrell, by Cappell, and by Lustig, and in the Soviet Union by the group around Mischenko and, in particular, Kasparov. Of the recent results the most impressive were due to Farrell and Jones, but others, in particular Connes-Moscivici and Ferry-Weinberger, have also proved exciting theorems. Although this work had surgery in the background, it did not, except for the work of Cappell, rely on new breakthroughs in surgery theory. In this sense it differed from the third period.

At the same time new and more powerful surgery theories, bounded and controlled surgery, were being explored to deal with similar sorts of topological questions. In an important paper Quinn solved the problem: When is a space that locally homotopically looks like a manifold actually a manifold? His methods involved a fine control of surgery obstructions of local character, and this opened a path to the development of controlled surgery. At the same time the related but somewhat coarser notion of bounded topology was initiated, motivated by the study of infinite group actions on $R^n$. The idea here was to get topological conditions to permit one to compactify effectively group actions at $\infty$. This was the strategy Mostow so brilliantly exploited in his geometric context. This led naturally to a notion of bounded surgery. Finally, influenced by both developments, Weinberger began an ambitious program to classify stratified spaces and developed a stratified surgery theory which is close to Quinn's conception. All these developments are relatively recent and not fully worked out. They suggest that surgery will continue to play a technically and calculationally significant role in HDGT but without the dominating force of the third period.

It is at this moment in surgery's history that DS appears. It is concerned with a problem that arose in the third period of HDGT.

One of the crucial and most useful features of classical surgery theory is the fourfold periodicity of the surgery obstruction groups; that is, the groups $L_n(\ )$ depend only on $n$ mod 4. Further, this periodicity is realized by multiplication of a surgery problem by $\mathrm{CP}^2$, the complex projective space of real dimension 4. One would like to have this property for the equivariant surgery. The investigation of this question is the main task of DS.

For theories of the Browder-Quinn type there is no problem. The classical proof goes through directly. For theories of Petrie type there is a basic problem; namely, multiplication by $\mathrm{CP}^2$, with the trivial group action, does not preserve the gap hypothesis. To get around this, many investigators have followed up a suggestion Browder made in 1976: multiply instead by products of $\mathrm{CP}^2$'s but with nontrivial action.

The work that Dovermann-Schultz do in this volume is to pursue Browder's suggestion in a number of different contexts. They show that this method works provided one assumes various side conditions which are too technical to elaborate in this review. The central structure of their proofs is standard in this context, although the detailed argument is complicated and at points delicate. One reduces the equivariant obstruction to a sequence of obstructions, each of which is closely tied to a classic obstruction, and then uses strong forms of classical periodicity.

These forms of classical periodicity are all based on product formulae, i.e., what happens when one multiplies a surgery problem by a given closed manifold. This itself is one of the most difficult and delicate questions of classical surgery theory. As we have noted, the understanding of what happens when taking products with tori was crucial to the results of the second period of HDGT. Dovermann-Schultz need extensions of a strong form of such formulae due to Yoshida, who studies such questions for an important extension of classical surgery theory due to Cappell-Shaneson, surgery with coefficients. This part is the technical heart of this work.

As a result of this work, we can now conclude that forms of periodicity hold in a large number of cases for a variety of equivariant surgery theories. A satisfactory general formulation is not achieved, and one can legitimately be a bit disappointed here. Still, as a research monograph, this is a worthwhile achievement, and its results have interesting applications, some of which are pointed out by the authors. It also provides for the specialist in this area the most detailed available discussion of the relationship between the various equivariant surgery theories.

The serious student of such periodicity phenomena will also want to supplement this book by taking into account some related fourth-period developments. Particularly the results on the more delicate question of periodicity in the structure sets, of Siebenmann in the nonequivariant case, and of Weinberger-M. Yan in the equivariant case seem clearly related.

The reader of DS should be forewarned that the background demands are formidable. Not only is one expected to be familiar with the book of Wall, but one is also expected to be familiar with the details of the long and difficult publications of Browder-Quinn, Dovermann-Petrie, Dovermann-Rothenberg, Luck-Madsen, Ranicki and Yoshida. While some of this work is briefly summarized in DS, a more detailed and leisurely summary, perhaps in appendices, would have made the book more accessible.

Overall, my assessment is that this volume will find a useful and respected place in the libraries of specialists in surgery theory. While not for the casual reader, DS will provide the intrepid reader quite a bit of information about equivariant surgery groups and their periodicity properties. The techniques developed will surely prove useful in future investigations of periodicity questions and their applications to transformation group theory. However, a real doubt remains in my mind as to whether fundamental breakthroughs in HDGT lie in this direction. In any case, the book that will attain the broader audience that the subject, due to the simplicity, beauty, power, and historical influence of its core ideas, deserves remains to be written.

An extended bibliography would be out of place in a book review. Below is a list which allows the industrious reader to run down the original papers of the results referred to plus a few selected works I find enlightening on the matters discussed.

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MEL ROTHENBERG
UNIVERSITY OF CHICAGO

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 2, April 1993

0273-0979/93 $1.00 + $.25 per page

*Pluripotential theory*, by Maciej Klimek. London Math. Soc. Monographs (N.S.), vol. 6, Oxford Univ. Press, New York, 1991, xiv+266 pp., $59.95. ISBN 0-19-853568-6

We will first discuss *potential* theory in one complex variable and then the higher-dimensional *pluripotential* theory.

At first, recall the concept of a harmonic function.

Holomorphic functions in the complex plane satisfy the Cauchy Riemann equation $\partial f/\partial \overline{z} = 0$. One sees immediately that the two-dimensional Laplacian

$$\partial^2/\partial x^2 + \partial^2/\partial y^2 =: \Delta = 4\partial/\partial z \partial/\partial \overline{z}.$$

Hence holomorphic functions are also solutions of the homogeneous Laplace equation. In fact both the real and imaginary parts of a holomorphic function are solutions to the Laplace equation. In general, in any dimension, the solutions of the homogeneous Laplace equation are called harmonic functions. A special property in one complex dimension is that the real-valued harmonic functions on the complex plane are the real parts of holomorphic functions.

In $\mathbb{R}^3$ the solutions of the homogeneous Laplace equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0$$

are called harmonic functions or *Newtonian potentials*. Their gradient is a vectorfield called *force*, and specific examples are given by *gravitational potential* and *gravitational force* or *electric potential* and *electric force*. The basic example