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AN EXTERNAL APPROACH TO UNITARY REPRESENTATIONS

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INTRODUCTION

The principal ideas of harmonic analysis on a locally compact group G which is not necessarily compact or commutative were developed in the 1940s and early 1950s. In this theory, the role of the classical fundamental harmonics is played by the irreducible unitary representations of G. The set of all equivalence classes of such representations is denoted by $\widehat{G}$ and is called the dual object of G or the unitary dual of G.

Since the 1940s, an intensive study of the foundations of harmonic analysis on complex and real reductive groups has been in progress (for a definition of reductive groups, the reader may consult the appendix at the end of §2). The motivation for this development came from mathematical physics, differential equations, differential geometry, number theory, etc. Through the 1960s, progress in the direction of the Plancherel formula for real reductive groups was great, due mainly to Harish-Chandra's monumental work, while at the same time, the unitary duals of only a few groups had been parametrized.

With Mautner's work [Ma], a study of harmonic analysis on reductive groups over other locally compact nondiscrete fields was started. We shall first describe such fields. In the sequel, a locally compact nondiscrete field will be called a local field.

If we have a nondiscrete absolute value on the field $\mathbb{Q}$ of rational numbers, then it is equivalent either to the standard absolute value (and the completion is the field $\mathbb{R}$ of real numbers) or to a p-adic absolute value for some prime number p. For $r \in \mathbb{Q}^\times$ write $r = p^\alpha a/b$ where α, a, and b are integers and neither a nor b are divisible by p. Then the p-adic absolute value of r is

$$|r|_p = p^{-\alpha}.$$

A completion of $\mathbb{Q}$ with respect to the p-adic absolute value is denoted by $\mathbb{Q}_p$. It is called a field of p-adic numbers. Each finite-dimensional extension F of $\mathbb{Q}_p$ has a natural topology of a vector space over $\mathbb{Q}_p$. With this topology, F becomes a local field. The topology of F can also be introduced with an absolute value which is denoted by $|\ |_F$ (in §5 we shall fix a natural absolute value). The fields of real and complex numbers, together with the finite extensions of p-adic numbers, exhaust all local fields of characteristic zero up to isomorphisms [We].

Let $\mathbb{F}$ be a finite field. Denote by $\mathbb{F}((X))$ the field of formal power series over $\mathbb{F}$. Elements of this field are series of the form $f = \sum_{n=k}^{\infty} a_n X^n$, $a_n \in \mathbb{F}$, for some integer k. Fix $q > 1$. Very often q is taken to be the cardinal number of the finite field $\mathbb{F}$. One defines an absolute value of f by the formula

$$|f|_{\mathbb{F}((X))} = q^{-\min\{n\,;\,a_n \neq 0\}}$$

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when $f \neq 0$. In this way $\mathbb{F}((X))$ becomes a local field. Fields of formal power series over finite fields exhaust all local fields of positive characteristic up to isomorphisms [We].

The fields $\mathbb{R}$ and $\mathbb{C}$ are called archimedean fields. For any $x, y \in \mathbb{R}^\times$ or $x, y \in \mathbb{C}^\times$, there is always a positive integer n such that $|y| < |nx|$. The above property does not hold for any other local field. This is the reason that local fields which are not isomorphic to $\mathbb{R}$ or $\mathbb{C}$ are called nonarchimedean local fields.

After Mautner in the 1960s, a series of people started to consider reductive groups over nonarchimedean local fields. Let us recall that p-adic fields were introduced historically to enable one to consider a single equation over a p-adic field instead of an infinite series of congruences mod p^k. Arithmetical problems also provided motivation to consider representations of reductive groups over such fields. The strongest motivation comes from the Langlands program. A unifying element in this program is the representation theory of reductive groups. A nice introduction to the Langlands program is [Gb3].

Let G be a reductive group over a local field. Harish-Chandra created a strategy for obtaining the unitary dual $\widehat{G}$ through the nonunitary dual $\widetilde{G}$, where $\widetilde{G}$ is the set of all functional equivalence classes of topologically completely irreducible continuous representations of G. Functional equivalence means that the matrix coefficients of one representation may be approximated by matrix coefficients of another on compact sets, and vice versa. A complete definition of $\widetilde{G}$ is in §2. To obtain $\widehat{G}$, one needs to classify $\widetilde{G}$ (the problem of the nonunitary dual) and to identify $\widehat{G} \subseteq \widetilde{G}$ (the unitarizability problem). In [L2] Langlands showed how to parametrize $\widetilde{G}$ by irreducible representations with certain good asymptotic properties (tempered representations) of reductive subgroups, when the field F is $\mathbb{R}$. The tempered representations were classified for $F = \mathbb{R}$ by Knapp and Zuckerman [KnZu] on the basis of Harish-Chandra's work, thus providing a complete picture of $\widetilde{G}$. Despite the Langlands classification of $\widetilde{G}$ in the archimedean case, there were no big breakthroughs in the classification of unitary duals for quite a long time. Borel-Wallach and Silberger proved that Langlands parametrization of $\widetilde{G}$ in terms of tempered representations of reductive subgroups was valid for reductive groups over all local fields [BlWh, Si1].

In this paper, we shall be concerned with the unitarizability problem for reductive groups over local fields. One usually breaks the unitarizability problem into two parts. The first part is constructing elements of $\widehat{G}$, and the second is showing that the constructed representations exhaust $\widehat{G}$ (completeness argument). The completeness argument is usually realized by showing that the classes of $\widetilde{G}\backslash\widehat{G}$ are not unitarizable. We may call such an approach to the completeness argument indirect.

Suppose that the field is archimedean. Then one can linearize the problems for $\widehat{G}$ (and $\widetilde{G}$); one can "differentiate" the representations and come to the infinitesimal theory where the main object is the Lie algebra $\mathfrak{g}$ of G. Also, for a maximal compact subgroup K of G, the theory of compact Lie groups gives an explicit description of $\widehat{K}$. Thus, one may try to understand $(\pi, H) \in \widetilde{G}$ by studying the restriction of π to K. These two points explain why, in the archimedean case, some problems concerning representations, and especially the unitarizability problem, were often approached by studying the internal structure of representations. Let us recall that the internal approach was very successful in the compact Lie group case (restriction to a maximal torus). In the nonarchimedean case, there is no possibility of such an internal approach to the unitarizability problem. One of the reasons that there has been much less study of the unitarizability problem is that the nonunitary duals are not yet completely parametrized there.

Despite the fact that unitary duals of a very restricted number of groups have been classified, it is interesting to note that in 1950 Gelfand and Naimark published a book [GfN2] in which they constructed what they assumed to be the dual objects of the complex classical simple Lie groups. Their lists were very simple, and the representations were also simple (although infinite dimensional). Gelfand and Naimark were using functional analytic methods as tools in their analysis. In 1967 Stein constructed, in a fairly simple manner, representations in $\mathrm{GL}(2n, \mathbb{C})\hat{}$ which were not contained in the lists of Gelfand and Naimark [St]. For some other classical groups, it was even easier to see the incompleteness of the lists from [GfN2].

The representations of Gelfand and Naimark of $\mathrm{GL}(n, \mathbb{C})$, complemented by Stein, were not generally expected to exhaust the whole of $\mathrm{GL}(n, \mathbb{C})\hat{}$.

The main aim of this paper is to present the ideas which lead first to the solution of the unitarizability problem for $\mathrm{GL}(n)$ over nonarchimedean local fields [Td3] and to the recognition that the same result holds over archimedean local fields [Td2], a result which was proved by Vogan [Vo3] using an internal approach. Let us say that the approach that we are going to present may be characterized as external. At no point do we go into the internal structure of representations.

Let us present the answer. We fix a general local field F. Let D^u be the set of all functional equivalence classes of irreducible square integrable modulo center representations of all $\mathrm{GL}(n, F)$, $n \geq 1$ (for definition see §3). Let $|\ |_F$ be the modulus of F (see §5). For each representation $\delta \in D^u$ of $\mathrm{GL}(n, F)$, and for each $m \geq 1$, consider the representation of $\mathrm{GL}(mn, F)$ parabolically induced (§1) by

$$|\det|_F^{(m-1)/2}\delta \otimes |\det|_F^{(m-3)/2}\delta \otimes \cdots \otimes |\det|_F^{-(m-1)/2}\delta$$

from a suitable standard parabolic subgroup (i.e., from one containing the upper triangular matrices, §2). This representation has a unique irreducible quotient which will be denoted by $u(\delta, m)$. For $0 < \alpha < 1/2$ let $\pi(u(\delta, m), \alpha)$ be the representation of $\mathrm{GL}(2mn, F)$ parabolically induced by

$$|\det|_F^{\alpha} u(\delta, m) \otimes |\det|_F^{-\alpha} u(\delta, m).$$

Denote by B all possible $u(\delta, m)$ and $\pi(u(\delta, m), \alpha)$. Then the answer is

Theorem. (i) *Let* $\tau_1, \ldots, \tau_n \in B$. *Then the representation* π *parabolically induced by*

$$\tau_1 \otimes \cdots \otimes \tau_n$$

of suitable $\mathrm{GL}(m, F)$ *is irreducible and unitary.*

(ii) *Suppose that* ρ *is obtained from* $\sigma_1, \ldots, \sigma_n \in B$ *in the same manner as* π *was obtained from* $\tau_1, \ldots, \tau_m$ *in* (i). *Then* $\pi \cong \rho$ *if and only if* $n = m$ *and the sequences* $(\tau_1, \ldots, \tau_n)$ *and* $(\sigma_1, \ldots, \sigma_n)$ *coincide after a renumeration.*

(iii) *Each irreducible unitary representation of* $\mathrm{GL}(m, F)$, *for any* m, *can be obtained as in* (i).

A new, and at the same time very old, point of view that led to the papers [Td1–Td3] was that the unitarizability problem has a reasonably simple answer and that the unitary representations appear in simple and natural ways. In the completeness argument, instead of an indirect strategy, a direct argument was used. In this way, a detailed study of nonunitary representations was avoided. In all arguments, essentially only Hilbert space representations were necessary.

Surprisingly, the statement of the theorem, which was first discovered in the nonarchimedean case in [Td3], says that in the case $F = \mathbb{C}$ the unitary dual of $\mathrm{GL}(n, \mathbb{C})$ should consist of the representations of Gelfand, Naimark, and Stein. Not only the statement of the nonarchimedean case of the theorem, but also the methods of the proof in [Td3] made sense in the archimedean case. Thus, after a complete proof had been written in the nonarchimedean case, we wrote in [Td2] the proof of the archimedean case of the theorem. We have used there a theorem of Kirillov from [Ki1], for which he never published the complete proof (see §9).

There are now many solutions of the unitarizability problems, especially for particular reductive groups in the archimedean case. In general, they are based on ideas different than the one that we present in this paper. Two of them take a distinguished place (they solve completely the problem for a series of groups having no bounds on their semisimple split ranks). The first is Vogan classification in [Vo3] of the unitary duals of $\mathrm{GL}(n)$ over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. He proved a theorem equivalent to the statement of our theorem for archimedean F. The other one is Barbasch's classification of the unitary duals of the complex classical Lie groups in [Bb].

Since we consider both the archimedean and nonarchimedean cases, it is natural to recall Harish-Chandra's Lefschetz principle: "Whatever is true for real reductive groups is also true for p-adic groups" [Ha2]. One problem with the Lefschetz principle is that we usually obtain a result for archimedean F by one kind of considerations and for nonarchimedean F by very different methods, and after that we compare the results. An interesting problem is to explain the phenomenon of the Lefschetz principle, which is certainly related to our depth of understanding of these two theories. An important task of this paper is to present a unified point of view on the theorem and its proof. We shall discuss both the theorem and the proof, making no distinction concerning the nature of the field. This is possible by the use of the external point of view. In our approach we shall very often be close to the point of view of the representation theory of general locally compact groups, and this will be a unifying point.

Professor P. J. Sally suggested that I write a paper where the ideas would be extracted from the technical machinery as much as possible. I am thankful to him for his advice and encouragement and for his generous help during the writing of various drafts of this paper, which he has read. I am also thankful for the hospitality and support of the University of Chicago during the summer of 1986 when a sketch of this paper was prepared. I would like to express my thanks to a number of mathematicians for helpful discussions or encouragement during the writing of this paper and the preceding ones on the same topic.

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We hope that this paper will illustrate a certain internal symmetry in the external approach to the unitarizability problem. We hope that some of our ideas will be helpful in dealing with the unitary duals of other nonarchimedean classical groups (and also nonunitary duals). The papers [SlTd] and [Td8] indicate that this hope is not without basis.

Finally, we introduce some general notation which we shall use throughout the paper. For a topological space X, $C(X)$ will denote the space of all continuous (complex-valued) functions on X. The subspace of all compactly supported continuous functions will be denoted by $C_c(X)$. If we have a measure μ on X, then $L^1(X, \mu)$ will denote the space of all classes of μ-integrable functions on X and $L^2(X, \mu)$ will denote the Hilbert space of all classes of square integrable functions on X with respect to μ. For a smooth manifold X the space of smooth functions on X will be denoted by $C^\infty(X)$ and $C^\infty(X) \cap C_c(X)$ will be denoted by $C_c^\infty(X)$. If X is a totally disconnected locally compact topological space, then $C_c^\infty(X)$ will denote the space of all compactly supported locally constant functions on X. The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively. The ring of integers is denoted by $\mathbb{Z}$, nonnegative integers are denoted by $\mathbb{Z}_+$, and positive ones are denoted by $\mathbb{N}$.

1. Concept of harmonic analysis on general locally compact groups

In this section we shall outline some of the ideas of harmonic analysis on locally compact groups.

Let G be a locally compact group. We shall always suppose in this paper that the groups are separable. A *representation* of G is a pair (π, V) where V is a complex vector space which is not zero dimensional and π is a homomorphism of G into the group of all linear isomorphisms of V. By a *continuous representation* of G we shall mean a representation (π, H) of G where H is a Hilbert space and the map $(g, v) \mapsto \pi(g)v$, $G \times H \to H$ is continuous (we shall always assume that H is separable). A closed subspace H' of H will be called a *subrepresentation* of a continuous representation (π, H) of G, if H' is invariant for all operators $\pi(g)$, $g \in G$. A continuous representation (π, H) of G is called *irreducible* if there does not exist a nontrivial subrepresentation of (π, H) (i.e., different from $\{0\}$ and H). A continuous representation (π, H) of G will be called a *unitary representation* if all the operators $\pi(g)$, $g \in G$, are unitary. Two unitary representations (π_i, H_i), $i = 1, 2$, of G are called *unitarily equivalent* if there exists a Hilbert space isomorphism $\varphi : H_1 \to H_2$ such that $\pi_2(g)\varphi = \varphi\pi_1(g)$ for all $g \in G$. The set of all unitarily equivalence classes of irreducible unitary representations of G will be denoted by $\widehat{G}$ and called the *unitary dual* of G. For a family (π_i, H_i), $i \in I$, of unitary representations of G, there is a natural unitary representation of G on the direct sum of Hilbert spaces $\bigoplus_{i \in I} H_i$. This representation will be denoted by $(\bigoplus_{i\in I} \pi_i, \bigoplus_{i \in I} H_i)$. It is called a *direct sum* of representations (π_i, H_i), $i \in I$.

A continuous representation (resp. unitary representation) on a one-dimensional space is called a *character* (resp. *unitary character*) of G.

The main problem of harmonic analysis on the group G is to understand some interesting unitary representations of G (such representations are usually given on function spaces). One way to study this problem is to break it into two parts (at least for type I groups which will be described in the sequel of this section and which will be the only groups considered in this paper). The first part is to understand irreducible unitary representations, i.e., $\widehat{G}$, and the second part is to understand other unitary representations in terms of irreducible ones. This strategy is in the spirit of Fourier's classical idea of fundamental harmonics.

Let us explain what we mean by understanding general unitary representations in terms of irreducible ones. If G is a compact group, then a fundamental fact is that each unitary representation of G can be decomposed into a direct sum of irreducible unitary representations [Di, Theorem 15.1.3.]. Understanding a unitary representation π in terms of irreducible unitary representations means, in the compact case, to know how to decompose π into a direct sum of irreducible unitary representations. In the noncompact case each unitary representation decomposes into a direct integral of irreducible unitary representations, and understanding here again means to know how to decompose a given unitary representation into a direct integral of irreducible representations. We are not going to define here the notion of direct integral of representations because the definition is quite technical. The interested reader may consult [Di, §8]. Let me only mention that the direct integral generalizes the notion of direct sum and that direct integrals are determined by measures on $\widehat{G}$. To consider measures on $\widehat{G}$, one needs some σ-ring of sets. This σ-ring arises in a standard way from the natural topological structure on $\widehat{G}$. Now we shall define this topology.

If (π, H) is a continuous representation of G and $v, w \in H$, then the function $g \mapsto (\pi(g)v, w)$ on G is called a *matrix coefficient* of (π, H). We denote by $\Phi(\pi)$ the linear span of all matrix coefficients of (π, H). The closure operator on subsets of $\widehat{G}$ is defined as follows. Let $\pi \in \widehat{G}$ and $X \subseteq \widehat{G}$. Then $\pi \in \mathrm{Cl}(X)$ if and only if each element of $\Phi(\pi)$ can be approximated uniformly on each compact subset of G by elements from $\bigcup_{\sigma \in X} \Phi(\sigma)$. This topology on $\widehat{G}$ will be called the *topology of the unitary dual* of G.

If G is a commutative group, then each irreducible unitary representation of G is given on a one-dimensional space (this follows easily from the spectral theorem). Thus each $\pi \in \widehat{G}$ is a function. Pointwise multiplication of element of $\widehat{G}$ defines a group structure on $\widehat{G}$. Together with the above topology, $\widehat{G}$ is again a locally compact commutative group which is called the *dual group* of G. The role of the topology of the unitary dual is crucial in harmonic analysis on locally compact commutative groups. This topology is a basis on which one builds the fundamental facts of the harmonic analysis on commutative groups.

One of these fundamental facts is that G and $(G^\wedge)^\wedge$ are canonically isomorphic (Pontryagin duality).

In the case of noncommutative groups, the role of this topology on $\widehat{G}$ is less crucial than in the commutative case, but it is still an important and natural object to consider. For example, G is a *type I group* if and only if $\widehat{G}$ is a T_0-space, i.e., for any two different points in $\widehat{G}$, at least one of them has a neighborhood which does not contain the other one [Di, 9.5.2. and Theorem 9.1]. It is important to notice that, in general, $\widehat{G}$ is not topologically homogeneous and there exist significant connections between properties of irreducible representations and their position in $\widehat{G}$ with respect to the topology.

Before we proceed further, we shall say a few words about some measures which are natural to consider on locally compact groups. For a locally compact group G, there exists a positive measure invariant for right translations. Such a measure will be called a *right Haar measure* on G and it will be denoted by μ_G. Thus,

$$\int_G f(gx)\,d\mu_G(g) = \int_G f(g)\,d\mu_G(g)$$

for any $f \in C_c(G)$ and $x \in G$. Any two right Haar measures are proportional. The behavior of a right Haar measure for left translations is described by the *modular function*. There exists a function Δ_G on G such that

$$\int_G f(xg)\,d\mu_G(g) = \Delta_G(x)^{-1} \int_G f(g)\,d\mu_G(g)$$

for any $f \in C_c(G)$ and $x \in G$. The group G is called *unimodular* if $\Delta_G \equiv 1$, i.e., if μ_G is also invariant for left translations. For more information about Haar measures and for proofs of the above facts one may consult [Bu2].

Suppose that G is unimodular. The space $C_c(G)$ becomes an algebra for the convolution which is defined by the formula

$$(f_1 * f_2)(x) = \int_G f_1(xg^{-1})f_2(g)\,d\mu_G(g),$$

$f_1, f_2 \in C_c(G)$. In a natural way one can extend the convolution to $L^1(G, \mu_G)$. Then $L^1(G, \mu_G)$ becomes a Banach algebra. For $f \in C_c(G)$ and a continuous representation π of G set

$$\pi(f) = \int_G f(g)\pi(g)\,d\mu_G(g).$$

Now π becomes a representation of the convolution algebra $C_c(G)$, and this representation is called the *integrated form* of the representation π of G. If π is unitary, then the last formula also defines a representation of the algebra $L^1(G, \mu_G)$. Moreover, it is a $*$-representation if we define $f^*(g) = \overline{f(g^{-1})}$.

We have already mentioned that the basic problem of harmonic analysis on G is to classify $\widehat{G}$ and then to decompose interesting representations in terms of $\widehat{G}$. Among the interesting representations, there is one that should be the first to be understood, namely, the regular representation on $L^2(G, \mu_G)$. By the abstract Plancherel theorem, for a unimodular group G there exists a unique positive measure ν on $\widehat{G}$ such that

$$\int_G |f(g)|^2\,d\mu_G(g) = \int_{\widehat{G}} \mathrm{Trace}(\pi(f)\pi(f)^*)\,d\nu(\pi)$$

for all $f \in L^1(G, \mu_G) \cap L^2(G, \mu_G)$ [Di, Theorem 18.8.2]. The measure ν is called the *Plancherel measure* of G (it determines an explicit decomposition of $L^2(G, \mu_G)$ into a direct integral of elements of $\widehat{G}$).

While the basic ideas of harmonic analysis on general locally compact groups were laid down in the 1940s, one of the first breakthroughs in classifying unitary duals was the work of Kirillov for nilpotent Lie groups at the beginning of the 1960s (see [Ki2, §§13 and 15]).

Let us first recall that a *Lie group* is a group supplied with a structure of a (real) analytic manifold such that the group operations (i.e., multiplication and inversion) are analytic mappings. We can define the *Lie algebra* $\mathfrak{g}$ *of a Lie group* G as the tangent space of G at the identity, supplied with a bracket operation $[\ ,\ ]$ which can be defined in the following way. If $X, Y \in \mathfrak{g}$ are tangent vectors to curves $x(t)$, $y(t)$ for $t = 0$ respectively, then $[X, Y]$ is the tangent vector to the curve

$$t \mapsto x(\tau)y(\tau)x(\tau)^{-1}y(\tau)^{-1}$$

where $\tau = \operatorname{sgn}(t)|t|^{1/2}$, at $t = 0$ [Ki2, 6.3.]. An element $g \in G$ acts on G by inner automorphism. The differential of this action is denoted by $\mathrm{Ad}(g)$. In this way G acts on $\mathfrak{g}$, and this action is called the *adjoint action* of G on $\mathfrak{g}$.

Let G be a connected simply connected nilpotent Lie group, and let $\mathfrak{g}^*$ be the space of linear forms on the Lie algebra $\mathfrak{g}$ of G. There is a natural action of G on $\mathfrak{g}^*$. It is called the *coadjoint action* of G. Using the theory of induced representations (which we shall discuss a bit later in this section), Kirillov established a canonical one-to-one mapping from the set of all coadjoint orbits onto the unitary dual of G

$$G\backslash\mathfrak{g}^* \to \widehat{G}$$

which gives a simple description of the unitary dual. With a natural topology on the left-hand side, this is a homeomorphism. Kirillov theory also gives the characters of irreducible unitary representations and the Plancherel formula for nilpotent Lie groups.

In the second part of this section, we define some notions that we shall need in the sequel.

If G is a compact group, then each $\pi \in \widehat{G}$ is given on a finite-dimensional space [Di, 15.1.4.]. As in the theory of finite group representations, the function

$$\Theta_\pi : g \mapsto \operatorname{Trace} \pi(g),$$

which is called the *character of the representation* π, completely determines the class of π in $\widehat{G}$. We have a right Haar measure on G. Thus the character function determines a distribution on a compact Lie group G. It is easy to see that this distribution is

$$f \mapsto \operatorname{Trace} \pi(f),\ f \in C_c^\infty(G),$$

which will be denoted by Θ_π again. If π is an infinite-dimensional representation, obviously the trace as a function is not well defined. Nevertheless, it may happen that the above distribution is well defined. For an arbitrary Lie group G, one can take the above distribution for the definition of the character of $\pi \in \widehat{G}$, if $\pi(f)$ is trace-class for $f \in C_c^\infty(G)$. If $\pi(f)$ has a trace for

$f \in C_c^\infty(G)$, then $\pi(f)$ must be a compact operator for any $f \in L^1(G, \mu_G)$. If all $\pi \in \widehat{G}$ have characters in the above sense, then $\pi(f)$ is a compact operator for any $\pi \in \widehat{G}$ and $f \in L^1(G, \mu_G)$.

A locally compact group is called a *CCR-group* if $\pi(f)$ is a compact operator for any $\pi \in \widehat{G}$ and $f \in L^1(G, \mu_G)$. All CCR-groups are of type I [Di, Proposition 4.3.4. and Theorem 9.1]. A great number of very important groups are CCR-groups. The most important classes of CCR-groups are commutative groups, compact groups, nilpotent Lie groups, and reductive groups over local fields (reductive groups will be defined in the appendix at the end of the following section). In particular, classical groups over local fields are CCR-groups. One can characterize CCR-groups in terms of the topology of the unitary duals. A group G is a CCR-group if and only if $\widehat{G}$ is a T_1-space, i.e., if the points are closed subsets of $\widehat{G}$ [Di, 9.5.3.].

For some important classes of groups one can show that they are CCR-groups by showing that they have so-called "large" compact subgroups. Now we shall explain the last notion.

Let G be a locally compact unimodular group, and let K be a compact subgroup. For $\delta \in \widehat{K}$ and a continuous representation (π, H_π) of G, let $H_\pi(\delta)$ be the subspace of H_π spanned by all subrepresentations of $\pi|K$ which are isomorphic to δ (here $\pi|K$ denotes the representation of K obtained by restriction from G). If

$$\dim_{\mathbb{C}} H_\pi(\delta) < \infty$$

for all $\delta \in \widehat{K}$, then π is called a *representation of G with finite K-multiplicities.*

One calls K a *large compact subgroup* of G if for each $\delta \in \widehat{K}$ the function

$$\pi \mapsto \dim_{\mathbb{C}} H_\pi(\delta)$$

is a bounded function on $\widehat{G}$. If G has a large compact subgroup, then G is a CCR-group [Di, Theorem 15.5.2]. Some very important classes of groups have large compact subgroups, for example, reductive groups over local fields.

For a continuous representation (π, H) of G, a vector $v \in H$ is called *K-finite* if the span of all $\pi(k)v$, $k \in K$, is finite dimensional.

In the rest of this section we shall discuss some parts of the theory of the induced representations for locally compact groups. The notion of induced representations for locally compact groups generalizes the well-known notion of induced representations for finite groups that was introduced and studied by Frobenius and Schur. Induction is one of the simplest and most important procedures for obtaining new representations of locally compact groups.

The most important case of induction for the purpose of this paper is *parabolic induction.* To define this notion, it is enough to consider a closed subgroup P of a unimodular group G and assume that there exists a compact subgroup K of G such that $PK = G$. Let (σ, M) be a continuous representation of P. The space of all (classes of) measurable functions $f : G \to H$ which satisfy

$$f(pg) = \Delta_P(p)^{1/2}\sigma(p)f(g), \qquad p \in P,\ g \in G,$$

and

$$||f||^2 = \int_K ||f(k)||^2 \, d\mu_K(k) \ < \ \infty$$

will be denoted by $\mathrm{Ind}_P^G(\sigma)$. It is a Hilbert space. The group G acts by right translations on $\mathrm{Ind}_P^G(\sigma)$. This action we denote by R. Thus

$$(R_g f)(x) = f(xg).$$

With this action, $\mathrm{Ind}_P^G(\sigma)$ is a continuous representation of G. It is unitary if σ is unitary.

In our considerations G will be a reductive group over a local field, while P will be a parabolic subgroup of G (these terms will be defined in the following section). One will take a Levi decomposition $P = MN$ of P and a continuous representation σ of M. Since $P/N \cong M$, we shall consider σ as a representation of P. Then we shall say that $\mathrm{Ind}_P^G(\sigma)$ is a *parabolically induced representation* of G by σ.

We shall also talk at some points in this paper about induced representations which are of more general type. Let G be a locally compact group which does not need to be unimodular, and let C be a closed subgroup of G. Suppose that a unitary representation (σ, H) of C is given (we may assume that σ is a continuous representation only). We denote by $\mathrm{Ind}_C^G(\sigma)$ the space of all measurable functions $f : G \to H$ which satisfy

$$f(cg) = [\Delta_C(c)\Delta_G(g)^{-1}]^{1/2}\sigma(c)f(g), \qquad g \in G,\ c \in C.$$

One square integrability condition is required also. This condition is more technical than in the case of parabolic induction [Ki2, 13.2]. Again G acts by right translations on $\mathrm{Ind}_C^G(\sigma)$. This is a unitary representation of G. We say that $\mathrm{Ind}_C^G(\sigma)$ is *unitarily induced* by σ.

Mackey obtained a simple criterion for testing if a given unitary representation of G is unitarily induced from C (Imprimitivity Theorem, [Ki2]). There is an important specialization of this theory. If G contains a nontrivial normal abelian subgroup N, then Mackey theory implies a description of $\widehat{G}$ by irreducible unitary representations of smaller groups (G and N need to satisfy certain general topological conditions). Let $\chi \in \widehat{N}$ and denote by G_χ the stabilizer of χ in G (G acts on $\widehat{N}$ because G acts on N by automorphisms). Note that $\widehat{N}$ consists of unitary characters. Take an irreducible unitary representation σ of G_χ such that $\sigma|N$ is a multiple of χ. Then $\mathrm{Ind}_{G_\chi}^G(\sigma)$ is an irreducible unitary representation of G. One obtains all irreducible unitary representations of G by the above construction. Two such representations constructed from χ_1, σ_1 and χ_2, σ_2 are equivalent if and only if χ_1, σ_1 and χ_2, σ_2 are conjugate. This specialization of Mackey theory is usually called *small Mackey theory.* In the special case when G is a semidirect product of a closed normal abelian subgroup N and a closed subgroup M, the small Mackey theory describes $\widehat{G}$ more simply. Here $\widehat{G}$ is parametrized by unitary characters of N and irreducible unitary representations of their stabilizers in M, divided by a natural equivalence.

For our purpose, the case of semidirect products is the most interesting one. It is important to observe that one obtains here automatically the irreducibility of some induced representations. For a more detailed exposition of small Mackey theory, one may consult [Ki2, 13.3].

2. The nonunitary dual as a tool for the unitary dual

In the study of the representation theory of the general linear groups, and more generally, of the classical groups, the terminology of the theory of reductive groups is very useful and natural. We shall very briefly recall some of the terminology of this theory in the first part of the appendix at the end of this section. The reader may also skip over the general definitions and follow only the case of $\mathrm{GL}(n)$ where we shall not need these general definitions.

We begin this section with a few definitions. We shall first define parabolic subgroups in $\mathrm{GL}(n, F)$. Take an ordered partition

$$\alpha = (n_1, n_2, \dots, n_k)$$

of n. Consider block-matrices

$$A = \begin{bmatrix} A_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & A_{1k} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{k1} & \cdot & \cdot & \cdot & \cdot & \cdot & A_{kk} \end{bmatrix}$$

where A_{ij} is an n_i by n_j matrix. Denote

$$P_\alpha = \{A \in \mathrm{GL}(n, F);\, A_{ij} = 0 \text{ for } i > j\},$$

$$M_\alpha = \{A \in \mathrm{GL}(n, F);\, A_{ij} = 0 \text{ for } i \neq j\}.$$

Let N_α be the set of all $A \in P_\alpha$ such that all A_{ii} are identity matrices. Now P_α is called a standard parabolic subgroup of $\mathrm{GL}(n, F)$, M_α is called a Levi factor of P_α, and N_α is called the unipotent radical of P_α. The subgroup $P_{(1,1,\dots,1)}$ of all upper triangular matrices in $\mathrm{GL}(n, F)$ is called the standard minimal parabolic subgroup.

Take any $g \in \mathrm{GL}(n, F)$ and any ordered partition $\alpha = (n_1, n_2, \dots, n_k)$ of n. Then $gP_\alpha g^{-1}$ is called a parabolic subgroup of $\mathrm{GL}(n, F)$. Set $P' = gP_\alpha g^{-1}$, $M' = gM_\alpha g^{-1}$, and $N' = gN_\alpha g^{-1}$. Then $P' = M'N'$ is called a Levi decomposition of P'. The group M' is called a Levi factor of P', and N' is called the unipotent radical of P'. Similarly, a minimal parabolic subgroup is defined to be any conjugate of the standard minimal parabolic subgroup.

We shall now introduce the nonunitary dual. We denote by G a reductive group over a local field F. There exists a maximal compact subgroup K of G such that $P_{\min}K = G$ for some minimal parabolic subgroup $P_{\min}$ of G. We fix such a maximal compact subgroup. The Iwasawa decomposition $P_{\min}K = G$ holds for any maximal compact subgroup K of G if F is an archimedean field. If it is not, this may not be true for all maximal compact subgroups. The group K is a large compact subgroup of G in the sense of the previous section. In the case of $\mathrm{GL}(n, F)$ and $F = \mathbb{R}$ (resp. $F = \mathbb{C}$), one may take for K the group of orthogonal matrices (resp. the group of unitary matrices). If F is nonarchimedean, one may take K in $\mathrm{GL}(n, F)$ to be $\mathrm{GL}(n, \mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers in F, that is, $\mathcal{O}_F = \{x \in F;\, |x|_F \leq 1\}$. In a general linear group over any local field, all maximal compact subgroups are conjugate.

We shall always assume in the sequel that continuous representations of G that we consider have finite K-multiplicities.

For a continuous representation π of G, we have denoted by $\Phi(\pi)$ the linear span of all matrix coefficients of π. This is a subspace of $C(G)$. Denote by $\mathrm{Cl}\,\Phi(\pi)$ the closure of $\Phi(\pi)$ with respect to the open-compact topology on

$C(G)$. Then two continuous representations π_1 and π_2 of G are said to be *functional equivalent* if

$$\mathrm{Cl}\,\Phi(\pi_1) = \mathrm{Cl}\,\Phi(\pi_2).$$

We denote by $\widetilde{G}$ the set of all functional equivalence classes of continuous irreducible representations of G which have finite K-multiplicities. These representations are precisely the topologically completely irreducible representations of G (for a definition of the last notion, one may consult [Wr, 4.2.2.]). The set $\widetilde{G}$ is called the *nonunitary dual* of G. We could use Naimark equivalence to define $\widetilde{G}$ instead of the functional equivalence. We would get the same object. Two continuous representations (π_1, H_1) and (π_2, H_2) are called *Naimark equivalent* if there exist dense subspaces $V_1 \subseteq H_1$ and $V_2 \subseteq H_2$ which are invariant for integrated forms and a closed one-to-one linear operator φ from V_1 onto V_2 such that

$$(\pi_2(f)\varphi)(x) = (\varphi\pi_1(f))(x)$$

for any $f \in C_c(G)$, $x \in V_1$.

For an irreducible continuous representation π of G which has finite K-multiplicities, the operator $\pi(f)$ is of trace-class if $f \in C_c^\infty(G)$. The linear form

$$f \mapsto \mathrm{Trace}(\pi(f))$$

is denoted by Θ_π and called the *character of the representation* π. Two irreducible continuous representations with finite K-multiplicities are functionally equivalent if and only if they have the same characters. Furthermore, characters of representations from $\widetilde{G}$ are linearly independent.

The natural mapping $\widehat{G} \to \widetilde{G}$ is one-to-one (G is a CCR-group). Therefore, we shall identify $\widehat{G}$ with a subset of $\widetilde{G}$. A class $\pi \in \widetilde{G}$ will be called *unitarizable* or *unitary* if $\pi \in \widehat{G} \subseteq \widetilde{G}$. One supplies $\widetilde{G}$ with a topology in the same way as we have supplied $\widehat{G}$ with the topology of the unitary dual.

The reader may consult the second part of the appendix at the end of this section for standard realizations of the set $\widetilde{G}$. Those realizations depend on whether the field is archimedean or not.

The idea of Harish-Chandra was to break the problem of describing $\widehat{G}$ into two parts: the problem of the nonunitary dual and the unitarizability problem. The problem of determining the nonunitary dual appeared to be much more manageable than the unitarizability problem.

Besides the above general strategy, there could be other strategies for getting $\widehat{G}$. It would be interesting to obtain a classification of the unitary duals dealing with nonunitary representations as little as possible or perhaps not at all. A strategy of such classification for $\mathrm{GL}(n)$ over archimedean fields dealing with only unitary representations can be based on a paper from 1962 by Kirillov [Ki1]. We shall return to [Ki1] in §9. Now we shall outline the strategy.

Let P_n be the subgroup of $\mathrm{GL}(n, F)$ of all matrices with the bottom row equal to $(0, 0, \dots, 0, 1)$. An interesting property of P_n follows from the small Mackey theory: $\widehat{P}_n$ is in a bijection with

$$\mathrm{GL}(n-1, F)\hat{} \cup \mathrm{GL}(n-2, F)\hat{} \cup \cdots \cup \mathrm{GL}(2, F)\hat{} \cup \mathrm{GL}(1, F)\hat{} \cup \mathrm{GL}(0, F)\hat{}$$

($\mathrm{GL}(0, F)$ denotes the trivial group). Gelfand and Naimark showed already in 1950 that certain irreducible unitary representations of $\mathrm{SL}(n, \mathbb{C})$, which they expected to exhaust $\mathrm{SL}(n, \mathbb{C})^\wedge$, remain irreducible as representations of P_n' where P_n' denotes the subgroup of $\mathrm{SL}(n, \mathbb{C})$ of all matrices with the bottom row of the form $(0, \dots, 0, x)$, $x \in \mathbb{C}^\times$. They obtained this result from the explicit formulas for those representations. Kirillov's idea was to first find a general proof of the irreducibility of $\pi|P_n$ for $\pi \in \mathrm{GL}(n, F)^\wedge$ when $F = \mathbb{R}$ or $\mathbb{C}$. Then one has an inductive procedure for classification. After classifying $\mathrm{GL}(m, F)^\wedge$ for $m \leq n-1$, one also has a classification of $\widehat{P}_n$ by the above remark about $\widehat{P}_n$. Thus, the second part of the strategy is to find all possible extensions of representations from $\widehat{P}_n$ to unitary representations of $\mathrm{GL}(n, F)$. This strategy was used by I. J. Vahutinskii in his study of irreducible unitary representations of $\mathrm{GL}(3, \mathbb{R})$.

At the end of this section we shall say a few words about the characteristics of some approaches to the unitary duals of certain groups in two relatively simple cases.

We shall first consider the case of a compact Lie group G. We shall assume that G is connected. One starts from an irreducible unitary representation (π, H) of G, and studying the internal structure of H, one comes to exact parameters which classify $\widehat{G}$. Let us give a rough idea of how this approach goes. A closed connected commutative subgroup of G is called a *torus* in G. Each torus is isomorphic to some $\mathbb{T}^n$ where $\mathbb{T}$ is the group of all complex numbers of norm one. We fix a maximal torus T in G. Then each element of G is conjugate to an element of T, i.e., each conjugacy class of G intersects T. Suppose that π is a continuous representation of G on a finite-dimensional space H. One can choose an inner product on H invariant for the action of G. Thus the representation $\pi|T$ decomposes into a direct sum of some unitary characters $\chi_1, \dots, \chi_m$. These unitary characters are called the *weights* of π with respect to T. Take now $(\pi, H) \in \widehat{G}$. We have already mentioned that the character Θ_π of the representation π determines the class of π. Since the character Θ_π, as a function on G, is obviously constant on conjugacy classes, π is already determined by $\pi|T$, i.e., π is determined by its weights. Further analysis of the structure of the representation π on H requires the study of the representation of the Lie algebra (the formula (L.A.) in the appendix at the end of this section, defines the action of the Lie algebra). It gives that there is a particular weight among all weights of π which already characterizes π (the highest weight). In this way one obtains a parametrization of $\widehat{G}$ by a certain subset of characters of T (the dominant weights). For an exposition of this nice theory of Weyl and Cartan, one may consult, among many nice expositions, the seventh paragraph of [Bu1].

We shall present now a simple strategy for solving the unitarizability problem for $G = \mathrm{SL}(2, \mathbb{R})$. One may take for K the group $\mathrm{SO}(2)$ of all two-by-two orthogonal matrices of determinant one. Note that $\mathrm{SO}(2) \cong T$. The unitary dual of K is given by the characters

$$\sigma_n : \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \mapsto e^{in\varphi},$$

when n runs over $\mathbb{Z}$. If $(\pi, H) \in \widetilde{G}$, then it is not hard to show that multiplicities of the representation $\pi|K$ are one, i.e., $H(\sigma_n)$ are either zero- or one-dimensional subspaces. This can be seen in a similar way as one shows that the space of K-invariant vectors in an irreducible representation is one dimensional if (G, K) is a Gelfand pair [GfGrPi, Chapter III, §3, no. 4]. Thus, there is a basis $\{v_n ; n \in S\}$ of the Hilbert space H, parametrized by a subset S of $\mathbb{Z}$. Suppose that π is a unitary representation with a G-invariant inner product $(\ , \ )$ on H. The formula (L.A.) from the appendix defines the representation of the Lie algebra $\mathfrak{g}$ of G on K-finite vectors. Differentiating the relation $(\pi(g)v , \pi(g)w) = (v , w)$, $g \in G$, along one-parameter subgroups, one gets that the representation on K-finite vectors satisfies

$$(\pi(X)v , w) = -(v , \pi(X)w)$$

for all $X \in \mathfrak{g}$. Put $\|v_n\| = c_n$. Since v_n, $n \in S$, are orthogonal, the inner product is completely determined by numbers c_n, $n \in S$. One can solve the unitarizability problem in the following way. Take $(\pi , H) \in \widetilde{G}$ and check if there exist positive numbers c_n, $n \in S$, such that the inner product

$$\left(\sum \lambda_i v_i , \sum \mu_i v_i\right) = \sum c_i^2 \lambda_i \bar{\mu}_i$$

satisfies $(\pi(X)v , w) + (v , \pi(X)w) = 0$ for all K-finite vectors v and w in H and all $X \in \mathfrak{g}$. All π for which there exist such numbers form the unitary dual. Clearly, to be able to solve the above problem, we need to know explicitly the internal structure of irreducible representations of G.

In the above two examples, one solves the problem by study of the internal structure of representations.

Appendix

Algebraic groups. We shall recall very briefly in the first part of this appendix some definitions from the theory of algebraic groups. For precise definitions containing all details, one should consult [Bl].

A *linear algebraic group* $\mathbf{G}$ is a Zariski closed subgroup of some general linear group with entries from an algebraically closed field. A linear algebraic group is called *unipotent* if it is conjugate to a subgroup of the upper triangular unipotent matrices. A linear algebraic group $\mathbf{G}$ is called *reductive* if $\mathbf{G}$ does not contain a normal Zariski closed unipotent subgroup of positive dimension. A linear algebraic group $\mathbf{G}$ is called *semisimple* if it is reductive and if it has a finite center. If a noncommutative linear algebraic group $\mathbf{G}$ does not contain a proper normal Zariski closed subgroup of positive dimension, then $\mathbf{G}$ is called a *simple algebraic group.* If there is a group isomorphism of $\mathbf{G}$ onto some $\mathrm{GL}(1)^n$ which is also an isomorphism of algebraic varieties, then $\mathbf{G}$ is called a *torus.*

A Zariski closed subgroup $\mathbf{P}$ of $\mathbf{G}$ is called a *parabolic subgroup* if the homogeneous space $\mathbf{G}/\mathbf{P}$ is a projective variety.

By a *reductive group*, we shall mean the group $\mathbf{G}(F)$ of all F-rational points of a reductive group $\mathbf{G}$ which is defined as an algebraic variety over a local field F. By a *parabolic subgroup* of a reductive group $\mathbf{G}(F)$, we shall mean the group of all rational points of a parabolic subgroup of $\mathbf{G}$, which is defined over F. A *minimal parabolic subgroup* of $\mathbf{G}(F)$ is a parabolic subgroup which does not contain any other parabolic subgroup. All minimal parabolic subgroups in $\mathbf{G}(F)$ are conjugate. If we fix a minimal parabolic subgroup $P_{\min}$ of $\mathbf{G}(F)$, then the parabolic subgroups containing $P_{\min}$ are called *standard parabolic subgroups.*

Each parabolic subgroup in $\mathbf{G}(F)$ is conjugate to a standard parabolic subgroup. Let P be a parabolic subgroup of $\mathbf{G}(F)$. Among Zariski connected normal unipotent subgroups of P there is the maximal one. It is called the *unipotent radical* of P. Denote it by N. There exists a reductive subgroup M of P such that $P = MN$ and $M \cap N = \{1\}$. Note that P is then a semidirect product of N and M. One says that $P = MN$ is a *Levi decomposition* of P. Also, one says that M is a *Levi factor* of P.

Since a reductive group $\mathbf{G}(F)$ is a closed subgroup of $\mathrm{GL}(n, F)$, $\mathbf{G}(F)$ is in a natural way a locally compact group. If F is an archimedean field (i.e., $F = \mathbb{R}$ or $\mathbb{C}$), then $\mathbf{G}(F)$ is a Lie group in a natural way. If F is a nonarchimedean field, then $\mathbf{G}(F)$ is a totally disconnected group.

In the sequel, a reductive group $\mathbf{G}(F)$ will usually be denoted simply by G.

The most important examples of reductive groups are the classical groups such as general linear groups $\mathrm{GL}(n, F)$, special linear groups, symplectic groups and orthogonal groups. The groups $\mathrm{GL}(n, F)$ form the simplest series of reductive groups and one of the first series to be considered.

Even if one is interested in harmonic analysis on some particular class of classical groups, it is usually necessary to study a broader class of groups because some important constructions involve subgroups which may not belong to the considered class. Such subgroups are reductive. This is the reason why it is convenient to use the terminology of reductive groups even if we consider some specific class of groups.

Realizations of $\widetilde{G}$. The set $\widetilde{G}$ has the following realizations, depending on whether the field is archimedean or not. We shall not use these realizations in the sequel, so the reader can also skip over these definitions. We note that the following notions are very important in the theory. It is also interesting to note how different $\widetilde{G}$ looks in the following archimedean and nonarchimedean realizations.

Suppose that F is an archimedean field. First, we shall give a definition of a $(\mathfrak{g}, K)$-module ($\mathfrak{g}$ is the Lie algebra of G). A *representation* π *of a Lie algebra* $\mathfrak{g}$ is a real-linear map from $\mathfrak{g}$ into the space of all linear operators on a complex vector space V such that

$$\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$$

for any $X, Y \in \mathfrak{g}$. If we have a Lie group G and a continuous representation π of G on a finite-dimensional space V, then the following limit exists

$$\text{(L.A.)} \qquad \pi(X)v = \frac{d}{dt}[\pi(x(t)v)]_{t=0}, \qquad v \in V,$$

and it defines a representation of the Lie algebra $\mathfrak{g}$ of G on V (in the above formula $X \in \mathfrak{g}$ and X is the tangent vector to the curve $x(t)$ at $t = 0$). We call this Lie algebra representation the *differential* of π. Suppose now that G is a reductive group over F and $\mathfrak{k}$ the Lie algebra of K. Let (π, V) be a pair where V is a complex vector space and $\pi = (\pi_{\mathfrak{g}}, \pi_K)$ is again a pair consisting of a Lie algebra representation $\pi_{\mathfrak{g}}$ of $\mathfrak{g}$ on V and of a representation π_K of K on V, such that the following three conditions are satisfied.

(a) For each $v \in V$ the vector space W spanned by all $\pi_K(k)v$, $k \in K$, is finite dimensional and the representation of K on W is continuous.

(b) The differential of the representation π_K of K equals the restriction of the Lie algebra representation $\pi_{\mathfrak{g}}$ to $\mathfrak{k}$.

(c) For any $k \in K$, $X \in \mathfrak{g}$, and $v \in V$

$$\pi_{\mathfrak{g}}(\mathrm{Ad}(k)X)v = \pi_K(k)\pi_{\mathfrak{g}}(X)\pi_K(k^{-1})v.$$

Then (π, V) is called a $(\mathfrak{g}, K)$*-module*. An *irreducible* $(\mathfrak{g}, K)$*-module* is a $(\mathfrak{g}, K)$-module which has no nontrivial subspaces invariant both for actions of K and $\mathfrak{g}$. Two $(\mathfrak{g}, K)$-modules (π', V') and (π'', V'') are *equivalent* if there is a one-to-one linear mapping φ from V' onto V'' such that $\varphi\pi'_K(k) = \pi''_K(k)\varphi$ and $\varphi\pi'_{\mathfrak{g}}(X) = \pi''_{\mathfrak{g}}(X)\varphi$ for any $k \in K$ and $X \in \mathfrak{g}$. Now $\widetilde{G}$ is in a natural bijection with the set of all equivalence classes of irreducible $(\mathfrak{g}, K)$-modules. If (π, H) is an irreducible continuous representation of G (with finite K-multiplicities, which we always assume), then one takes for V the space of all K-finite vectors in H. The formula (L.A.) defines an action of $\mathfrak{g}$ on V, and V becomes an irreducible $(\mathfrak{g}, K)$-module in this way.

Suppose now that F is nonarchimedean. A representation (π, V) of G is called *smooth* if for each $v \in V$ there is an open subgroup K_v of G such that $\pi(k)v = v$ for any $k \in K_v$. Again we say that smooth representation (π, V) is *irreducible* if there is no nontrivial vector subspace invariant for the action of G. Two smooth representations (π_1, V_1) and (π_2, V_2) of G are *equivalent* if there exists a one-to-one linear map φ from V_1 onto V_2 such that $\pi_2(g)\varphi = \varphi\pi_1(g)$ for any $g \in G$. As before, there is a natural one-to-one correspondence from $\widetilde{G}$ onto the set of all equivalence classes of irreducible smooth representations of G. If (π, H) is an irreducible continuous representation of G, one takes again for V the space of all K-finite vectors v in H. Now the restriction of the action of G on H to V defines an irreducible smooth representation of G on V.

3. Some simple constructions of unitary representations

One would like to have rather simple and natural constructions of unitary representations which produce the whole of $\widehat{G}$. For nilpotent Lie groups such a systematic procedure consists of unitary induction by one-dimensional unitary representations. For the groups we consider, the situation is not so simple, but it is not too bad either. For example, one obtains the whole of $\mathrm{SL}(n, \mathbb{C})\widehat{\ }$ by parabolic induction with one-dimensional, in general nonunitary, representations (see §9).

We have introduced the topology of $\widehat{G}$ in §1. In the construction of new unitary representations, the hardest problem is to find new connected components of $\widehat{G}$. Since $\widehat{G}$ is not topologically homogeneous, there may exist special connected components, those consisting of only one point. These representations are usually called *isolated representations.* To avoid the influence of the commutative harmonic analysis coming from $G^{\mathrm{ab}} = G/G^{\mathrm{der}}$ where G^{der} denotes the derived group of G, we shall define the notion of representations isolated modulo center. Let $Z(G)$ be the center of G. For $(\pi, H) \in \widetilde{G}$ there exists a character $\omega_\pi \in Z(G)\widetilde{\ }$ such that $\pi(z) = \omega_\pi(z)\,\mathrm{id}_H$ for all $z \in Z(G)$. The character ω_π is called the *central character* of π. For a character $\omega \in Z(G)\widetilde{\ }$

set

$$\widetilde{G}_\omega = \{\pi \in \widetilde{G};\ \omega_\pi = \omega\} \quad \text{and} \quad \widehat{G}_\omega = \widetilde{G}_\omega \cap \widehat{G}.$$

The representation $\pi \in \widehat{G}$ (resp. $\pi \in \widetilde{G}$) will be called *isolated modulo center* (resp. *isolated modulo center in the nonunitary dual*) if $\{\pi\}$ is an open subset of $\widehat{G}_{\omega_\pi}$ (resp. an open subset of $\widetilde{G}_{\omega_\pi}$). In the sequel, by isolated representation we shall mean isolated modulo center. According to what we have said about the topology of $\widehat{G}$, we may say roughly that matrix coefficients of isolated representations are not similar to other matrix coefficients of elements of $\widehat{G}$. The following example indicates that. If G is compact, then $\widehat{G}$ is discrete, and matrix coefficients of different representations are L^2-orthogonal. Kazhdan proved in [Ka] that the trivial representation is isolated when G is a simple group of split rank $n \geq 2$. The *split rank* is the highest n such that G possesses a Zariski closed subgroup defined over F which is isomorphic over F to $\mathrm{GL}(1, F)^n$. As opposed to the trivial representation, other isolated representations are usually not easily constructible. In fact, isolated representations of $\widehat{G}$ or $\widetilde{G}$ are very distinguished representations in known examples. Certainly, each isolated representation in the nonunitary dual, which is unitary, is also isolated in the unitary dual.

The first isolated representations that one usually meets in the representation theory of reductive groups are square integrable. An irreducible unitary representation (π, H) of G is called *square integrable modulo center* if for any $v, w \in H$, the function

$$g \mapsto |(\pi(g)v, w)|$$

is a square integrable function on $G/Z(G)$ with respect to a Haar measure. We shall use the term *square integrable* instead of *square integrable modulo center*. Actually, the unitarity of some $(\pi, H) \in \widetilde{G}$ may be obtained from the above square integrability condition (note that the unitarity of the central character of π must be assumed to be able to formulate the above square integrability condition). The square integrable representations are crucial for Plancherel measure and important for parametrizing the nonunitary dual. In known examples, they are very often isolated (in the unitary dual).

The known examples show that the construction of a connected component, or at least a big part of it, reduces to the construction of isolated representations of reductive subgroups of G attached to parabolic subgroups and some standard simple and well-known constructions. Now we shall recall these standard simple constructions. The first and the oldest one is:

(a) *Unitary parabolic induction.* Let $P = MN$ be a Levi decomposition of a parabolic subgroup P of G. For a continuous representation σ of M, we have considered σ also as a representation of P using the projection $P = MN \to P/N \cong M$. Then $\mathrm{Ind}_P^G(\sigma)$ was called a parabolically induced representation of G. If σ is a unitary representation, then this process will be called unitary parabolic induction. In fact, we always take $\sigma \in \widehat{M}$. Then $\mathrm{Ind}_P^G(\sigma)$ is a unitary representation which is a direct sum of finitely many irreducible representations. It is usually irreducible. In the construction (a), we shall always assume that P is a proper subgroup of G. In general, if $(\ ,\ )$ is an M-invariant hermitian form on the representation space U of σ, then

$$(f_1, f_2) \mapsto \int_K (f_1(k), f_2(k))\, d\mu_K(k)$$

is a G-invariant hermitian form on $\mathrm{Ind}_P^G(\sigma)$, and it is positive definite if the form on U was positive definite.

Unitary parabolic induction has been used since the first classification of the unitary duals of reductive groups [Bg], [GfN1]. Gelfand and Naimark started to use systematically unitary parabolic induction for the classical simple complex groups, while Harish-Chandra started a systematic study of this induction.

The following construction was used also in the first classifications of unitary duals of reductive groups [Bg, GfN1].

(b) *Complementary series.* It happens that some representations induced by nonunitary ones become unitary after a new inner product is introduced on the representation space. The idea is the following. One realizes on the same space a "continuous" family (π_α, H_α), $\alpha \in X$, of irreducible induced representations which possess G-invariant nontrivial hermitian forms. Let X be connected. Suppose that some π_α is unitary. The fact that a continuous family of nondegenerate hermitian forms on a finite-dimensional space parametrized by X, being positive definite at one point of X, must be positive definite everywhere enables one to conclude that all constructed representations are in $\widehat{G}$ (here one reduces arguments to finite-dimensional spaces by considering spaces $\bigoplus H(\delta)$, where δ runs over fixed finite subsets of $\widehat{K}$). Positivity at one point is obtained in general from (a). The delicate point is the construction of a continuous family of G-invariant hermitian forms, and it is based on the theory of intertwining operators for induced representations.

For the above construction some authors use the term *deformation* [Vo4]. We have chosen rather a more traditional name.

Let us recall that a topological space X is quasi-compact if each open covering of X contains a finite open subcovering. Note that in the above definition of a quasi-compact topological space, the Hausdorff property is not required (this is the difference between quasi-compactness and compactness). A topological space is locally quasi-compact if each point has a fundamental set of neighborhoods which are quasi-compact. The fundamental fact about the topology of $\widehat{G}$ (actually, about the dual of any C^*-algebra) is local quasi-compactness. This fact essentially, together with some understanding of the topology of the unitary dual, implies that $\widehat{G}$ cannot be complete without

(c) *Irreducible subquotients of ends of complementary series.* This fact was first observed and proved by Miličić. Before we give a brief argument why the representations in (c) must be included in $\widehat{G}$, we shall give a simple but suggestive example.

Let P be a minimal parabolic subgroup in $G = \mathrm{GL}(2, F)$ (one may take for P the upper triangular matrices in G). We have denoted by Δ_P the modular character of P. Representations

$$I^\alpha = \mathrm{Ind}_P^G(\Delta_P^\alpha), \qquad -1/2 < \alpha < 1/2$$

are irreducible. If $\alpha = 0$, then I^0 is unitary by (a) (Δ_P^0 is the trivial representation, so it is unitary). The family I^α, $-1/2 < \alpha < 1/2$, is a "continuous" family of irreducible representations with nondegenerate G-invariant hermitian forms. Thus, they belong to $\widehat{G}$ by (b).

We shall pay attention now to the representation at the end of these complementary series $I^{-1/2} = \mathrm{Ind}_P^G(\Delta_P^{-1/2})$. From the definition of $I^{-1/2}$, it follows that the trivial representation of G is a subrepresentation of $I^{-1/2}$ (the trivial representation of some group G is a representation on a one-dimensional space V where each element acts as the identity on V). This is the unique proper nontrivial subrepresentation of $I^{-1/2}$. Since $I^{-1/2}$ is infinite dimensional, there is no inner product on $I^{-1/2}$ for which $I^{-1/2}$ is a unitary representation (in a unitary representation for each subrepresentation there is another subrepresentation on the orthogonal complement). Nevertheless, the representation on the quotient of $I^{-1/2}$ by the trivial representation is unitary (actually, it is square integrable, and this implies that it is unitary). So, though $I^{-1/2}$ is not unitary, each irreducible subquotient of the representation at the end of the complementary series is unitary. This is the case in general.

The representation $I^{-1/2}$ from the above considerations is a representation which is not irreducible, but also, it is not very far from being irreducible. This is an example of a representation of finite length. Before we proceed further with explanation of the construction (c), we shall recall the definition of a representation of finite length. Suppose that we have a continuous representation (π, H) of a reductive group G, which has finite K-multiplicities. Then we say that π is of *finite length* if there exist subrepresentations

$$\{0\} = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = H$$

of H, such that the quotient representations of G on H_i/H_{i-1} are irreducible representations of G, for $i = 1, 2, \ldots, n$. Parabolic induction carries the continuous representations of M of finite length to the continuous representations of G of finite length.

Now we shall give a brief argument for the unitarity of representations in (c). We shall omit technical details. Suppose that we have a complementary series of representations π_α, $\alpha \in X$. We may consider the following situation. Let Y be a topological space with a countable basis of open sets, and let X be a dense subset of Y. Assume that to each $\alpha \in Y$ is attached a nontrivial continuous representation π_α of G of finite length such that the functions

$$\alpha \mapsto \Theta_{\pi_\alpha}(\varphi), \qquad Y \to \mathbb{C}$$

are continuous, for all φ from the space $C_{c,*}(G)$ of all continuous compactly supported functions which span a finite-dimensional space after translations by elements of K (left and right). Suppose also that the π_α are irreducible unitary representations for all $\alpha \in X$. Take any $\alpha \in Y$. Let (α_n) be a sequence in X converging to α. Miličić proved that in general a sequence (π_n) in $\widehat{G}$ has no accumulation points if and only if $\lim_n \Theta_{\pi_n}(\varphi) = 0$ for all φ [Mi, Corollary of Theorem 6]. Since $\Theta_{\pi_\alpha} \not\equiv 0$, (π_{α_n}) has a convergent subsequence. If S is the set of all limits of subsequences of (π_{α_n}) in $\widehat{G}$, then Miličić's description of the topology of $\widehat{G}$ says that there exist positive integers n_σ, $\sigma \in S$, such that

$$\lim_n \Theta_{\pi_{\alpha_n}}(\varphi) = \sum_{\sigma \in S} n_\sigma \Theta_\sigma(\varphi)$$

for all φ in $C_{c,*}(G)$ [Mi, Theorems 6 and 7]. Also, S is a discrete and closed subset of $\widehat{G}$. Thus

$$\Theta_{\pi_\alpha}(\varphi) = \sum_{\sigma \in S} n_\sigma \Theta_\sigma(\varphi)$$

for all φ in $C_{c,*}(G)$. Since the character of π_α is the sum of characters of its irreducible subquotients, one can obtain easily that S is precisely the set of all irreducible subquotients of π_α. Since $S \subseteq \widehat{G}$, each irreducible subquotient of π_α is unitary. Thus (c) provides unitary classes.

For a direct proof without use of the topology, one may consult [Td5]. The proof is based on the fact that a group of unitary operators on finite-dimensional Hilbert space is finite dimensional. One uses in the proof the fact that reductive groups have large compact subgroups.

While the constructions of (a) and (b) provide bigger continuous families of unitary representations, (c) provides smaller families, but they are often important in the construction of unitary representations. Representations obtained by constructions (a), (b), or (c) are never isolated. We shall describe now a simple construction found by Speh that may produce isolated representations. This construction is particularly useful when it is combined with some other constructions, for example, with (a), (b), and (c). Contrary to previous constructions, here one gets unitarity of representations of smaller groups from unitarity of representations of bigger groups. Before we describe the construction, we need a notion of hermitian contragradient.

For a continuous representation (π, H) of G, $\bar{\pi}$ will denote the *complex conjugate* of π. It is the same representation, but the Hilbert space is the complex conjugate of H. The *contragradient representation* of π will be denoted by $\tilde{\pi}$. It is the representation on the space of all continuous linear forms on H with the action $[\tilde{\pi}(g)f](v) = f(\pi(g^{-1})v)$. Set $\pi^+ = \bar{\tilde{\pi}}$. Then π^+ will be called *hermitian contragradient* of π. A continuous irreducible representation π will be called *hermitian* if π and π^+ are in the same class in $\widetilde{G}$. It is easy to see that all unitary representations are hermitian. In the classifications of $\widetilde{G}$ it is usually easy to check whether π is hermitian or not.

Let $P = MN$ be a proper parabolic subgroup of G. It is very easy to prove the following fact:

(d) *Unitary parabolic reduction.* If we have hermitian $\sigma \in \widetilde{M}$ such that $\mathrm{Ind}_P^G(\sigma)$ is irreducible and that its class in $\widetilde{G}$ is unitarizable, then the class of σ is unitarizable too.

We shall now give a rough argument explaining why (d) provides unitarizable representations. Let (σ, H) be an irreducible nonunitarizable hermitian representation of M. Then there is a nondegenerate M-invariant hermitian form ψ on H. Now H decomposes $H = H_+ \oplus H_-$ as a representation of $K \cap P$, where ψ is positive definite on H_+ and negative definite on H_-. Clearly, $H_+ \neq \{0\}$ and $H_- \neq \{0\}$. Note that $\mathrm{Ind}_P^G(\sigma)$ and $\mathrm{Ind}_{K\cap P}^K(\sigma|K \cap P)$ are isomorphic as representations of K and

$$\mathrm{Ind}_{K\cap P}^K(\sigma|K \cap P) \cong \mathrm{Ind}_{K\cap P}^K(H_+) \oplus \mathrm{Ind}_{K\cap P}^K(H_-).$$

Since unitary induction carries unitary representations to unitary, we see that the hermitian form induced by ψ is indefinite. This form is also G-invariant.

Since a G-invariant hermitian form on an irreducible representation is unique up to a scalar, we see that if $\mathrm{Ind}_P^G(\sigma)$ is irreducible, then it is not unitarizable.

Roughly speaking, the construction (d) enables one to construct sometimes from a component of $\widehat{M}_1$ a component of $\widehat{M}_2$, where $P_i = M_i N_i$ are two parabolic subgroups of G.

There are also some simple constructions based on the geometry of a group or groups. For example, if $\sigma_i \in \widehat{G}_i$, then $\sigma_1 \otimes \sigma_2 \in (G_1 \times G_2)\hat{}$ (and conversely). There are also irreducible unitary representations which appear already in the classification of the nonunitary dual—the square integrable ones.

It is interesting to ask which constructions must be added to (a)–(d) to generate the whole unitary dual of the classical groups, starting with square integrable representations. We shall see that for the first class, the case of $\mathrm{GL}(n, F)$, the constructions (a)–(d) are enough.

Remarks. (1) Fell introduced in [Fe] a notion of nonunitary dual space for arbitrary locally compact group. It is a topological space consisting of so-called linear system representations. We have studied $\widetilde{G}$ as a topological space in [Td6] if G is a reductive group over a nonarchimedean field F. It was shown that $\widetilde{G}$ coincides with Fell's nonunitary dual. The set $\widehat{G}$ is a closed subset of $\widetilde{G}$. A representation $\pi \in \widetilde{G}$ is isolated modulo center if and only if there is a nontrivial matrix coefficient which is compactly supported modulo center. Therefore, one may interpret Jacquet's subrepresentation theorem [Cs, Theorem 5.1.2] in the following way. Each element of $\widetilde{G}$ can be obtained as a subrepresentation of $\mathrm{Ind}_P^G(\sigma)$, with σ isolated modulo center representation of M for some parabolic subgroup $P = MN$ of G. Each isolated modulo center representation of G in $\widetilde{G}$ is essentially unitary (i.e., it becomes unitary after twisting by a suitable character of G); actually it is essentially square integrable. Certainly, all these facts about the topology of the nonunitary dual should hold over archimedean fields with the proofs along the same lines. In the archimedean case, the representations with matrix coefficients compactly supported modulo center can exist only if $G/Z(G)$ is compact. Construction of representations of such groups is essentially solved by the case of compact Lie groups.

(2) Suppose that F is nonarchimedean. Let $P = MN$ be a parabolic subgroup in G. Let $\sigma \in \widetilde{M}$ be isolated modulo center. Denote by ${}^0G$ the set of all $g \in G$ such that the absolute value of $\mu(g)$ is one for any homomorphism $\mu : G \to F^\times$ which is also a morphism of algebraic varieties defined over F. Then a character $\chi : G \to \mathbb{C}^\times$ is called *unramified* if χ is trivial on ${}^0G$, i.e., if $\chi({}^0G) = \{1\}$. Let $U(M)$ be the set of all unramified characters of M. Then $U(M)\sigma = \{\chi\sigma;\ \chi \in U(M)\}$ is a connected component of $\widetilde{M}$ containing σ, and the set of all irreducible subquotients of $\mathrm{Ind}_P^G(\tau)$, $\tau \in U(M)\sigma$, is a connected component of $\widetilde{G}$. All connected components of $\widetilde{G}$ are obtained in this way [Td6]. So, for $\widetilde{G}$ the set of connected components reduces to the set of isolated representations modulo center in the nonunitary dual of the reductive parts of parabolic subgroups. Note that a difficult problem in the nonarchimedean case is the construction of representations isolated modulo center (i.e., of supercuspidal representations).

4. Completeness argument

In the last section, we have outlined constructions (a)–(d) of unitary representations of a reductive group G. It seems that those constructions provide a remarkable part of the unitary duals of the classical groups. This leads to one of the most interesting questions about unitary representations of reductive groups: How can one conclude that a set $X \subseteq \widehat{G}$ is a significant piece of $\widehat{G}$, or even all of it? At the present time there is no satisfactory strategy for answering such density questions. Recall the simple answer for finite groups: One needs to check if the sum of squares of degrees of representations in X is equal to the order of the group or not.

Suppose that a set $X \subseteq \widehat{G}$ is constructed and suppose also that one expects that it is the whole unitary dual. If one wants to prove that, then a usual strategy has been to prove that in $\widetilde{G}\backslash X$ all representations are nonunitarizable. One checks for each representation in $\widetilde{G}\backslash X$ that it cannot be unitarizable considering various properties of that representation. The simplest properties that one can consider are: if the representation is hermitian, if it has bounded matrix coefficients, etc. The construction (d) can be used also for getting nonunitarity (from $\widetilde{M}$ to $\widetilde{G}$). The above strategy we shall call the indirect strategy (of proving completeness of a given set of unitary representations).

The indirect strategy becomes less satisfactory for groups of larger size. At the same time, the indirect strategy is not completely satisfactory from the point of view of harmonic analysis: the stress is not on unitary representations, which are of the principal interest, but on nonunitary ones. Actually, one needs a very detailed knowledge of the structure of representations in $\widetilde{G}\backslash\widehat{G}$, and the set $\widetilde{G}\backslash\widehat{G}$ is usually much, much larger than the set $\widehat{G}$. The indirect strategy does not develop directly the intuition about unitary representations. In dealing with $\widetilde{G}$, it is very useful to algebracize the situation (the algebraic description of $\widetilde{G}$ was presented in the appendix at the end of §2).

Later on we shall present a completeness argument for $\mathrm{GL}(n)$, dealing simultaneously with all $\mathrm{GL}(n)$ and having only one argument rather than various cases. This will be an example of the direct strategy of proving completeness.

In the following section we shall explain the sense in which the set of representations of Gelfand, Naimark, and Stein is "big".

5. On the completeness argument: the example of $\mathrm{GL}(n, \mathbb{C})$

We shall first introduce some general notation for the general linear group and then pass to the complex case.

In the first part of this section, F denotes any local field.

For $x \in F^\times$, there exists a number $|x|_F > 0$ such that

$$|x|_F \int_F f(xg)\, d_{\mathrm{a}}(g) = \int_F f(g)\, d_{\mathrm{a}}(g)$$

for all $f \in C_{\mathrm{c}}(G)$ ($d_{\mathrm{a}}(g)$ denotes an additive invariant measure on F). Set $|0|_F = 0$. Then $|\ |_F$ is called the modulus of F. Note that $|\ |_{\mathbb{R}}$ is the usual absolute value on $\mathbb{R}$, $|\ |_{\mathbb{C}}$ is the square of the usual absolute value on $\mathbb{C}$ (i.e., $|z|_{\mathbb{C}} = z\bar{z}$), while $|x|_{\mathbb{Q}_p} = |x|_p$ (see the Introduction). For $g \in \mathrm{GL}(n, F)$ set

$$\nu(g) = |\det(g)|_F.$$

Clearly, $\nu : \mathrm{GL}(n, F) \to \mathbb{R}^\times$ is a character.

For $n_1, n_2 \in \mathbb{Z}_+$, we have denoted by $P_{(n_1, n_2)}$ the parabolic subgroup of $\mathrm{GL}(n_1 + n_2, F)$ consisting of the elements $g = (g_{ij})$ for which $g_{ij} = 0$ when $i > n_1$ and $j \leq n_1$. Also we have denoted $M_{(n_1, n_2)} = \{(g_{ij}) \in P_{(n_1, n_2)} ; g_{ij} = 0 \text{ for } j > n_1 \text{ and } i \leq n_1\}$. Then

$$M_{(n_1, n_2)} \cong \mathrm{GL}(n_1, F) \times \mathrm{GL}(n_2, F)$$

is a Levi factor of $P_{(n_1, n_2)}$.

For two continuous representations τ_i of $\mathrm{GL}(n_i; F)$, $i = 1, 2$, we consider $\tau_1 \otimes \tau_2$ as a representation of $M_{(n_1, n_2)} \cong \mathrm{GL}(n_1, F) \times \mathrm{GL}(n_2, F)$ in a natural way. The mapping $mn \mapsto (\tau_1 \otimes \tau_2)(m)$, where $m \in M_{(n_1, n_2)}$ and $n \in N_{(n_1, n_2)}$, defines a representation of $P_{(n_1, n_2)}$. This representation of $P_{(n_1, n_2)}$ was again denoted by $\tau_1 \otimes \tau_2$. Thus

$$\tau_1 \otimes \tau_2 : \begin{bmatrix} g_1 & * \\ 0 & g_2 \end{bmatrix} \mapsto \tau_1(g_1) \otimes \tau_2(g_2)$$

for $g_1 \in \mathrm{GL}(n_1, F)$ and $g_2 \in \mathrm{GL}(n_2, F)$. Now the parabolically induced representation

$$\mathrm{Ind}_{P_{(n_1, n_2)}}^{\mathrm{GL}(n_1 + n_2, F)}(\tau_1 \otimes \tau_2)$$

will be denoted by $\tau_1 \times \tau_2$. It is a standard fact that $(\tau_1 \times \tau_2) \times \tau_3$ is isomorphic to $\tau_1 \times (\tau_2 \times \tau_3)$ (i.e., there exists a continuous intertwining which is invertible). This is a consequence of a general theorem on induction in stages

$$\mathrm{Ind}_{H_2}^{H_3}(\mathrm{Ind}_{H_1}^{H_2}(\sigma)) \cong \mathrm{Ind}_{H_1}^{H_3}(\sigma).$$

Therefore, it makes sense to write $\tau_1 \times \tau_2 \times \tau_3$.

Before we explain an important property of the operation $\times$, we need the notion of associate parabolic subgroups and associate representations. Suppose that we have two parabolic subgroups P_1 and P_2 of some reductive group G. If we have Levi decompositions $P_1 = M_1 N_1$, $P_2 = M_2 N_2$, and $w \in G$ such that $M_2 = w M_1 w^{-1}$, then we say that P_1 and P_2 are *associate parabolic subgroups.* Suppose that σ_1 and σ_2 are finite length continuous representations of M_1 and M_2 respectively, such that $\sigma_2(m_2) \cong \sigma_1(w^{-1} m_2 w)$ for all $m_2 \in M_2$. Then we say that σ_1 and σ_2 are *associate representations.* For a continuous representation (π, H) of G of finite length, consider a sequence of subrepresentations

$$\{0\} = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_k = H$$

of H where the quotient representations on H_i/H_{i-1} are irreducible representations of G, for $i = 1, 2, \ldots, k$. Denote by $R(G)$ a free $\mathbb{Z}$-module which has for a basis $\widetilde{G}$. We shall denote by J.H.(π) the formal sum of all classes in $\widetilde{G}$ of the representations H_i/H_{i-1}, $i = 1, 2, \ldots, k$. We shall consider J.H.$(\pi) \in R(G)$ and call it the *Jordan-Hölder series of* π. The element J.H.$(\pi) \in R(G)$ does not depend on the filtration H_i, $i = 0, 1, 2, \ldots, k$ as above (one can see that from the linear independence of characters of representations in $\widetilde{G}$). Suppose that $P_1 = M_1 N_1$ and $P_2 = M_2 N_2$ are associate parabolic subgroups. Let σ_1 and σ_2 be associate representations of M_1 and M_2 (as before, we consider σ_1 and σ_2 as representations of P_1 and P_2 respectively). A standard fact about parabolic induction from associate parabolic subgroups by associate representations is that representations $\mathrm{Ind}_{P_1}^{G}(\sigma_1)$ and $\mathrm{Ind}_{P_2}^{G}(\sigma_2)$ have the same characters, which implies that these two representations have the same Jordan-Hölder series. The formula for the character of a parabolically induced representation from a minimal parabolic subgroup, when $F = \mathbb{R}$, is computed in [Wr, Theorem 5.5.3.1]. A similar formula holds without assumption on parabolic subgroup. In the nonarchimedean case we have a similar situation.

Suppose that τ_1 and τ_2 are continuous representations of finite length of $\mathrm{GL}(n, F)$ and $\mathrm{GL}(m, F)$ respectively. The above fact about induction from associated parabolic subgroups by associate representations implies that $\tau_1 \times \tau_2$ and $\tau_2 \times \tau_1$ have the same Jordan-Hölder series.

Set $\mathrm{Irr}^u = \bigcup_{n \geq 0} \mathrm{GL}(n, F)\hat{}$. To solve the unitarizability problem for the $\mathrm{GL}(n, F)$-groups, one needs to determine Irr^u.

In the rest of this section we shall assume $F = \mathbb{C}$. Recall that $|\ |_{\mathbb{C}}$ is the square of the standard absolute value that we usually consider on $\mathbb{C}$.

Let $\chi_0 : \mathbb{C}^\times \to \mathbb{C}^\times$ be the character $x \mapsto x|x|_{\mathbb{C}}^{-1/2}$. Since each $\pi \in \mathrm{GL}(n, \mathbb{C})\hat{}$ has a central character and $\mathrm{GL}(n, \mathbb{C})$ is a product of the center and of $\mathrm{SL}(n, \mathbb{C})$, the restriction of representations from $\mathrm{GL}(n, \mathbb{C})$ to $\mathrm{SL}(n, \mathbb{C})$ gives a one-to-one mapping of

$$\mathrm{GL}(n, \mathbb{C})\hat{}_{\chi_0} \cup \mathrm{GL}(n, \mathbb{C})\hat{}_{\chi_0^2} \cup \cdots \cup \mathrm{GL}(n, \mathbb{C})\hat{}_{\chi_0^n}$$

onto $\mathrm{SL}(n, \mathbb{C})\hat{}$. Therefore, in order to understand $\mathrm{SL}(n, \mathbb{C})\hat{}$ it is enough to understand $\mathrm{GL}(n, \mathbb{C})\hat{}$ (and conversely). In the rest of this paper we shall deal only with GL-groups and interpret the Gelfand, Naimark, and Stein representations in terms of $\mathrm{GL}(n)$.

The first obvious irreducible unitary representations of $\mathrm{GL}(n, \mathbb{C})\hat{}$ are one-dimensional representations $\chi \circ \det_n$ where χ is a unitary character of $\mathbb{C}^\times$ and where $\det_n$ denotes the determinant homomorphism of $\mathrm{GL}(n)$. Gelfand and Naimark obtained also the following series of irreducible unitary representations

$$(\nu^{-\alpha}\chi) \times (\nu^{\alpha}\chi) = [\nu^{-\alpha}(\chi \circ \det_1)] \times [\nu^{\alpha}(\chi \circ \det_1)], \qquad \chi \in (\mathbb{C}^\times)\hat{}, \ 0 < \alpha < \tfrac{1}{2},$$

the complementary series representations for $\mathrm{GL}(2, \mathbb{C})$. These complementary series start from representations

$$\chi \times \chi = (\chi \circ \det_1) \times (\chi \circ \det_1), \qquad \chi \in (\mathbb{C}^\times)\hat{}.$$

The following unitary representations of $\mathrm{GL}(n, \mathbb{C})$ will be obtained by parabolic induction using the above representations. Gelfand and Naimark showed in [GfN2] that the unitary representations obtained by parabolic induction using the representations above are irreducible. They assumed that in this way one gets all irreducible unitary representations of $\mathrm{GL}(n, \mathbb{C})$, up to unitary equivalence.

We can interpret the above remarks in the following way. Set

$$B_0 = \{\chi \circ \det_n, \ [\nu^{-\alpha}(\chi \circ \det_1)] \times [\nu^{\alpha}(\chi \circ \det_1)]; \ \chi \in (\mathbb{C}^\times)\hat{}, \ n \in \mathbb{N}, \ 0 < \alpha < \tfrac{1}{2}\}.$$

Gelfand and Naimark showed that for $\tau_1, \ldots, \tau_k \in B_0$, the representations $\tau_1 \times \cdots \times \tau_k$ are in Irr^u. Their assumption was that in this way one can get any representation from Irr^u.

Stein showed that the Gelfand and Naimark complementary series representations for $\mathrm{GL}(2, \mathbb{C})$ are just the first of the complementary series representations which exist for all $\mathrm{GL}(2n, \mathbb{C})$ [St]. He showed that

$$[\nu^{-\alpha}(\chi \circ \underset{n}{\det})] \times [\nu^{\alpha}(\chi \circ \underset{n}{\det})] \in \mathrm{Irr}^u$$

also for $n \geq 2$, $0 < \alpha < 1/2$, $\chi \in (\mathbb{C}^\times)\hat{}$, and he showed that these representations were not obtained by Gelfand and Naimark. These complementary series start from representations

$$(\chi \circ \underset{n}{\det}) \times (\chi \circ \underset{n}{\det}), \qquad n \geq 2, \ \chi \in (\mathbb{C}^\times)\hat{},$$

which were already well known to Gelfand and Naimark.

Now it is natural to complete the Gelfand and Naimark list by the Stein complementary series representations. Therefore, put

$$B = \{\chi \circ \underset{n}{\det}, \ [\nu^{-\alpha}(\chi \circ \underset{n}{\det})] \times [\nu^{\alpha}(\chi \circ \underset{n}{\det})]; \ \chi \in (\mathbb{C}^\times)\hat{}, \ n \in \mathbb{N}, \ 0 < \alpha < \tfrac{1}{2}\}$$

(i.e., B is just B_0 completed by the Stein complementary series representations). Now using arguments similar to those of Gelfand and Naimark, one may conclude that for $\tau_1, \ldots, \tau_k \in B$ the representations $\tau_1 \times \cdots \times \tau_k$ are in Irr^u (see [Sh]). Let us denote by {G.N.S.} the set of all representations obtainable in this way.

We can explain now in what sense {G.N.S.} is big in Irr^u. It is easy to prove the following fact (and we shall prove it later):

(D) *For any $\pi \in \mathrm{Irr}^u$, there exist τ_1, $\tau_2 \in$ {G.N.S.} such that $\pi \times \tau_1$ and τ_2 have a composition factor in common.*

It is clear that (D) plays a role in the completeness argument.

Regarding unitary parabolic induction for $\mathrm{GL}(n, \mathbb{C})$, the simplest situation would be if it were always irreducible. This is what Gelfand and Naimark expected to hold in 1950:

(S1) *Unitary parabolic induction for $\mathrm{GL}(n, \mathbb{C})$ is irreducible, i.e.,*

$$\tau_1, \ \tau_2 \in \mathrm{Irr}^u \quad \textit{implies} \quad \tau_1 \times \tau_2 \in \mathrm{Irr}^u.$$

Let us suppose that (S1) holds. Because of (S1) and (D), for each $\pi \in \mathrm{Irr}^u$ there exist τ_1, $\tau_2 \in$ {G.N.S.} such that $\pi \times \tau_1 = \tau_2$. Thus, to obtain Irr^u, it is enough to know how representations from {G.N.S.} can be parabolically induced.

We shall call $\pi \in \widehat{G}$ *primitive* if there is no proper parabolic subgroup $P = MN$ and $\sigma \in \widehat{M}$ so that $\pi \cong \mathrm{Ind}_P^G(\sigma)$. Certainly, each $\pi \in \widehat{G}$ is unitarily equivalent to some $\mathrm{Ind}_P^G(\sigma)$ where $\sigma \in \widehat{M}$ is primitive. We have mentioned that if P_1 and P_2 are associate parabolic subgroups and σ_1, σ_2 associate representations, then $\mathrm{Ind}_{P_1}^G(\sigma_1)$ and $\mathrm{Ind}_{P_2}^G(\sigma_2)$ have the same Jordan-Hölder series. The simplest situation would be if the converse were true for σ_1 and σ_2 primitive (because of the induction in stages, it is necessary to assume that σ_1 and σ_2 are primitive). For $\mathrm{GL}(n, \mathbb{C})$ this would mean (having (S1) in mind):

(S2) *If both families $\tau_i \in \mathrm{GL}(n_i, \mathbb{C})\hat{}$, $i = 1, \ldots, n$, and $\sigma_j \in \mathrm{GL}(m_j, \mathbb{C})\hat{}$, $j = 1, \ldots, m$, consist of primitive representations, and if*

$$\tau_1 \times \cdots \times \tau_n = \sigma_1 \times \cdots \times \sigma_m,$$

then $m = n$, *and after a renumeration, the sequences* $(\tau_1, \ldots, \tau_n)$ *and* $(\sigma_1, \ldots \ldots, \sigma_m)$ *are equal* (*all* n_i, m_j *are assumed to be* ≥ 1).

We shall assume that this holds.
A very plausible hypothesis is:

(S3) *The Stein representations*

$$[\nu^{-\alpha}(\chi \circ \underset{n}{\det})] \times [\nu^{\alpha}(\chi \circ \underset{n}{\det})]; \qquad \chi \in (\mathbb{C}^{\times})\hat{}, \ n \in \mathbb{N}, \ 0 < \alpha < \frac{1}{2},$$

are primitive.

Certainly, if we assume (S3), then all elements of B are primitive. Now it is obvious that (D), together with (S1), (S2), and (S3), implies

$$\mathrm{Irr}^u = \{\text{G.N.S.}\}.$$

It remains to find a way to prove (S1), (S2), and (S3). Note that until now only (classes of) unitary representations were necessary, and statements (S1), (S2), and (S3) are essentially analytic. So, if one could prove (D), (S1), (S2), and (S3) dealing only with unitary representations, one would have a classification of $\mathrm{GL}(n, \mathbb{C})\hat{}$ completely in terms of unitary representations.

As we shall see, (D) is easy to prove using noncomplicated parts of the nonunitary theory. The following strategy for proving (S2) and (S3) simultaneously can be used. The set Irr^u can be embedded in a suitable ring which is factorial and where multiplication corresponds to parabolic induction. Then one can prove that elements of B are prime or close to being prime. This would imply (S2) and (S3). This strategy would also include nonunitary theory (but again, not the complicated parts).

Finally, we leave the discussion about (S1) for §9. Let us say that the first ideas for proving (S1) are due to Gelfand, Naimark, and Kirillov.

In the following sections we shall elaborate in more detail the above strategy and outline such a strategy for $\mathrm{GL}(n)$ over general local field F.

6. The nonunitary dual of $\mathrm{GL}(n, F)$

In this section, we state some basic facts about a parametrization of $\mathrm{GL}(n, F)^{\sim}$.

Besides Irr^u which was introduced in the previous section, we introduce

$$\mathrm{Irr} = \bigcup_{n \geq 0} \mathrm{GL}(n, F)^{\sim}.$$

The set of all classes of square integrable representations in Irr^u of all $\mathrm{GL}(n, F)$, $n \geq 1$, will be denoted by D^u. The set of all essentially square integrable representations will be denoted by D. More precisely,

$$D = \{(\chi \circ \det)\, \delta\, ;\ \chi \in \mathrm{GL}(1, F)^{\sim}, \quad \delta \in D^u\}.$$

For a set X, $M(X)$ will denote the set of all finite multisets in X. These are all unordered n-tuples, $n \in \mathbb{Z}_+$. This is an additive semigroup for the operation

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_m) = (a_1, \ldots, a_n, b_1, \ldots, b_m).$$

By W_F we shall denote the Weil group of F if F is archimedean and the Weil-Deligne group in the nonarchimedean case. For the purposes of this paper, it will not be essential to know exactly the definition of W_F. We denote by I the set of all classes of irreducible finite-dimensional representations of W_F. By the local Langlands conjecture for $\mathrm{GL}(n)$ (which generalizes local class field theory), there should exist a natural one-to-one mapping of Irr onto classes of semisimple representations of W_F, i.e., onto $M(I)$

$$\mathrm{Irr} \to M(I).$$

Under such a mapping, D should correspond to I. Thus, there should exist a parametrization of Irr by $M(D)$. Let us write one such parametrization.

Let $a = (\delta_1, \dots, \delta_n) \in M(D)$. We can write

$$\delta_i = \nu^{e(\delta_i)}\delta_i^u, \qquad e(\delta_i) \in \mathbb{R}, \ \delta_i^u \in D^u.$$

After a renumeration, we may assume $e(\delta_1) \geq e(\delta_2) \geq \cdots \geq e(\delta_n)$. The representation

$$\lambda(a) = \delta_1 \times \cdots \times \delta_n$$

has a unique irreducible quotient (possibly $\lambda(a)$ itself), whose class depends only on a [BlWh, Jc1]. Its multiplicity in $\lambda(a)$ is 1. We shall denote this class by $L(a)$. Now

$$a \mapsto L(a), \qquad M(D) \to \mathrm{Irr}$$

is a one-to-one mapping onto Irr. It is a version of the Langlands parametrization of the nonunitary duals of $\mathrm{GL}(n)$-groups. Certainly, for the existence of such a parametrization, it is crucial that the parabolic induction by square integrable representations is irreducible for $\mathrm{GL}(n)$.

The formula for the hermitian contragradient in the Langlands classification becomes

$$L((\delta_1, \dots, \delta_n))^+ = L((\delta_1^+, \dots, \delta_n^+)).$$

If we write $\delta_i = \nu^{e(\delta_i)}\delta_i^u(e(\delta_i) \in \mathbb{R}, \ \delta_i^u \in D^u)$, then

$$\delta_i^+ = (\nu^{e(\delta_i)}\delta_i^u)^+ = \nu^{-e(\delta_i)}\delta_i^u.$$

Also, it is easy to show that

$$\nu^\alpha L((\delta_1, \dots, \delta_n)) = L((\nu^\alpha\delta_1, \dots, \nu^\alpha\delta_n)), \qquad \alpha \in \mathbb{C}.$$

Let R_n be the free abelian group with basis $\mathrm{GL}(n, F)^\sim$, i.e., $R_n = R(\mathrm{GL}(n, F))$. For each finite-length continuous representation π of $\mathrm{GL}(n, F)$, we have denoted by $\mathrm{J.H.}(\pi)$ its Jordan-Hölder series which is an element of R_n. Set

$$R = \bigoplus_{n \geq 0} R_n.$$

Now R is a graded additive group which is free over Irr. Let

$$R_n \times R_m \to R_{n+m}$$

be the $\mathbb{Z}$-bilinear mapping defined on the basis Irr by

$$(\sigma, \tau) \mapsto \mathrm{J.H.}(\sigma \times \tau).$$

This defines a multiplication on R, which will be denoted again by $\times$. We have mentioned in the previous section that $\tau \times \sigma$ and $\sigma \times \tau$ have the same Jordan-Hölder sequences. This just means the commutativity of R. Since $(\tau_1 \times \tau_2) \times \tau_3 \cong \tau_1 \times (\tau_2 \times \tau_3)$ (§5), R is also associative. Certainly, $\{L(a); \ a \in$

$M(D)\}$ is a basis of R. It is a standard fact that $\{\lambda(a);\ \ a \in M(D)\}$ is a basis of R (i.e., standard characters form a basis of the group of all virtual characters). This means nothing else than the following fact which was first noticed by Zelevinsky in the nonarchimedean case [Ze, Corollary 7.5].

6.1. **Proposition.** *The ring R is a $\mathbb{Z}$-polynomial algebra over all essentially square integrable representations* (*i.e., over D*).

In particular, R is a factorial ring.

It is a natural question to ask if it is possible to relate the operation of summing representations of W_F with some operation on representations from Irr in the correspondence $a \mapsto L(a)$; i.e., is there a relation between $L(a+b)$ and representations $L(a)$, $L(b)$? The answer is very nice: $L(a+b)$ is always a subquotient of $L(a) \times L(b)$.

One may find more information about the Langlands classification on the level of general reductive groups in [BlWh]. For the Langlands philosophy one may consult [Gb3].

7. Heuristic construction

In this section we shall try to see what would be the part of the unitary dual for the GL(n)-groups over arbitrary local field F, generated by classical constructions (a)–(d) of §3. Our principle will be to expect a situation as simple as we could assume, bearing our evidence in mind.

So, let us start with D^u. The representation $\delta \times \delta$ is irreducible, so $\delta \times \delta \in \mathrm{Irr}^u$ by the construction (a). Then we have the complementary series which starts from $\delta \times \delta$:

$$[\nu^{\alpha}\delta] \times [\nu^{-\alpha}\delta], \qquad 0 < \alpha < 1/2.$$

This is an example of construction (b). We require $\alpha < 1/2$ to ensure that the induced representation is irreducible. At the end of complementary series $[\nu^{\alpha}\delta] \times [\nu^{-\alpha}\delta]$, there will be unitary irreducible subquotients. To be able to identify at least one, let us recall the relation mentioned at the end of §6; namely,

$$L(a+b) \textit{ is a subquotient of } L(a) \times L(b) \textit{ for } a, b \in M(D).$$

We shall assume that this holds in the rest of this section. Therefore, we have that $L((\nu^{1/2}\delta, \nu^{-1/2}\delta))$ is unitary since it is at the end of the above complementary series (construction (c)).

To proceed further, in order to be able to form a new complementary series, let us suppose that for general linear groups *unitary parabolic induction is irreducible.* Then one has $L((\nu^{1/2}\delta, \nu^{-1/2}\delta)) \times L((\nu^{1/2}\delta, \nu^{-1/2}\delta)) \in \mathrm{Irr}^u$, and further, one has a new complementary series

$$[\nu^{\alpha}L((\nu^{1/2}\delta, \nu^{-1/2}\delta))] \times [\nu^{-\alpha}L((\nu^{1/2}\delta, \nu^{-1/2}\delta))], \quad 0 < \alpha < 1/2.$$

At the end of this complementary series

$$\begin{aligned}&[\nu^{1/2}L((\nu^{1/2}\delta, \nu^{-1/2}\delta))] \times [\nu^{-1/2}L((\nu^{1/2}\delta, \nu^{-1/2}\delta))]\\ &\quad = L((\nu\delta, \delta)) \times L((\delta, \nu^{-1}\delta))\end{aligned}$$

is the unitary subquotient

$$L((\nu\delta, \delta, \nu^{-1}\delta, \delta))$$

by the assumption that $L(a+b)$ is a subquotient of $L(a) \times L(b)$. It is natural to ask if the above representation is primitive. But $L((\nu\delta, \delta, \nu^{-1}\delta, \delta))$ is a subquotient of

$$L((\nu\delta, \delta, \nu^{-1}\delta)) \times L((\delta)).$$

Note that $L((\nu\delta, \delta, \nu^{-1}\delta)) \otimes L((\delta))$ is hermitian. If we take the simplest possibility, that is, $L((\nu\delta, \delta, \nu^{-1}\delta)) \times L((\delta))$ irreducible, then the construction (d) implies that

$$L((\nu\delta, \delta, \nu^{-1}\delta)) \otimes L((\delta))$$

is unitary. Clearly then, $L((\nu\delta, \delta, \nu^{-1}\delta)) \in \mathrm{Irr}^u$.

At this point it is convenient to introduce some notation. Set

$$a(\gamma, n) = (\nu^{(n-1)/2}\gamma, \nu^{(n-3)/2}\gamma, \ldots, \nu^{-(n-1)/2}\gamma)$$

for $\gamma \in D$ and $n \in \mathbb{Z}^+$ and

$$u(\gamma, n) = L(a(\gamma, n)).$$

We shall write

$$\nu^\alpha(\delta_1, \ldots, \delta_n) = (\nu^\alpha\delta_1, \ldots, \nu^\alpha\delta_n).$$

Now we proceed further. We already have $u(\delta, 1)$, $u(\delta, 2)$, $u(\delta, 3) \in \mathrm{Irr}^u$. One considers a complementary series

$$[\nu^\alpha u(\delta, 3)] \times [\nu^{-\alpha} u(\delta, 3)], \quad 0 < \alpha < 1/2,$$

which starts from $u(\delta, 3) \times u(\delta, 3)$. At the end we have the representation

$$L(\nu^{1/2}a(\delta, 3)) \times L(\nu^{-1/2}a(\delta, 3)),$$

and we can identify one irreducible subquotient which is

$$L(\nu^{1/2}a(\delta, 3) + \nu^{-1/2}a(\delta, 3)).$$

It is a unitary subquotient by (c). Note that

$$\nu^{1/2}a(\delta, 3) + \nu^{-1/2}a(\delta, 3) = a(\delta, 4) + a(\delta, 2)$$

which means that $L(a(\delta, 4) + a(\delta, 2))$ is unitary. Further, this representation is a subquotient of

$$u(\delta, 4) \times u(\delta, 2) = L(a(\delta, 4)) \times L(a(\delta, 2)).$$

Suppose again that $u(\delta, 4) \times u(\delta, 2)$ is irreducible. Since it is unitary, $u(\delta, 4) \otimes u(\delta, 2)$ needs to be unitary by the construction (d). Therefore, $u(\delta, 4)$ will be unitary.

Now it is easy to conclude that the assumption

$$u(\delta, n) \times u(\delta, n-2) \in \mathrm{Irr}$$

leads to

$$u(\delta, n) \in \mathrm{Irr}^u.$$

In the case of $\mathrm{GL}(n, \mathbb{C})$, D^u is equal to $\mathrm{GL}(1, \mathbb{C})\hat{} = (\mathbb{C}^\times)\hat{}$. Then $u(\delta, n) = \delta \circ \det_n$. So, we have not obtained new unitary representations in this case.

8. Scheme of unitarity for $\mathrm{GL}(n)$

In the last section we have seen heuristically what some simple assumptions suggest. Now we shall write down some of those assumptions and their "implications". We first recall that

$$u(\delta, n) = L(a(\delta, n)) = L((\nu^{(n-1)/2}\delta, \nu^{(n-3)/2}\delta, \ldots, \nu^{-(n-1)/2}\delta))$$

for $\delta \in D$ and $n \geq 1$. It was also observed in §6 that R is a factorial ring.

We introduce the following statements:

(U0) $\tau, \sigma \in \mathrm{Irr}^u \Rightarrow \tau \times \sigma \in \mathrm{Irr}$.
(U1) $\delta \in D^u, \quad n \in \mathbb{N} \Rightarrow u(\delta, n) \in \mathrm{Irr}^u$.
(U2) $\delta \in D^u, \quad n \in \mathbb{N}, \quad 0 < \alpha < 1/2 \Rightarrow [\nu^\alpha u(\delta, n)] \times [\nu^{-\alpha} u(\delta, n)] \in \mathrm{Irr}^u$.
(U3) $\delta \in D, \quad n \in \mathbb{N} \Rightarrow u(\delta, n)$ is a prime element of R.
(U4) $a, b \in M(D) \Rightarrow L(a+b)$ is a composition factor of $L(a) \times L(b)$.

Here only (U3) was not assumed or obtained in the last section. It is a strengthening of the assumption that the $u(\delta, n)$'s are primitive, which was present in the last section (otherwise, we would have tried to construct new unitary representations in that way).

We have seen that (U0) and (U4) lead to (U1) and (U2) (i.e., unitarity of the representations mentioned there). But it is interesting and surprising that with the addition of only one assumption, namely, (U3), the preceding assumptions also easily imply completeness for the unitary duals of the groups $\mathrm{GL}(n, F)$.

8.1. **Proposition.** *Suppose that* (U0)–(U4) *hold true. Set*

$$B = \{u(\delta, n), \quad [\nu^\alpha u(\delta, n)] \times [\nu^{-\alpha} u(\delta, n)], \quad \delta \in D^u, \quad n \in \mathbb{N}, \quad 0 < \alpha < \tfrac{1}{2}\}.$$

Then:

(i) *if* $\tau_1, \dots, \tau_k \in B$, *we have* $\tau_1 \times \cdots \times \tau_k \in \mathrm{Irr}^u$;
(ii) *if* $\pi \in \mathrm{Irr}^u$, *then there exist* $\sigma_1, \dots, \sigma_m \in B$ *such that*

$$\pi = \sigma_1 \times \cdots \times \sigma_m;$$

(iii) *if* $\sigma_1, \dots, \sigma_k, \quad \tau_1, \dots, \tau_m \in B$ *and* $\sigma_1 \times \cdots \times \sigma_k = \tau_1 \times \cdots \times \tau_m$, *then* $k = m$ *and the sequences* $\sigma_1, \dots, \sigma_k$ *and* $\tau_1, \dots, \tau_m$ *coincide after a renumeration.*

From (U0), (U1), and (U2) one obtains (i) directly. Also (U3) implies (iii). It remains to prove only (ii). First we shall prove

8.2. **Lemma.** *Suppose that* $\pi \in \mathrm{Irr}$ *is hermitian. If* (U4) *holds, then there exist* $\sigma_1, \dots, \sigma_n, \quad \tau_1, \dots, \tau_m \in B$ *such that* $\pi \times \sigma_1 \times \cdots \times \sigma_n$ *and* $\tau_1 \times \cdots \times \tau_m$ *have a composition factor in common.*

Proof. Let $\delta \in D^u, \quad k \in (1/2)\mathbb{Z}_+$, and $0 < \beta < 1/2$. Then

$$(\nu^{-k}\delta, \nu^k\delta) + a(\delta, 2k-1) = a(\delta, 2k+1), \qquad k > 0,$$

and

$$\begin{aligned}(\nu^{-k-\beta}\delta, \nu^{k+\beta}\delta) &+ \nu^{\beta-1/2}a(\delta, 2k) + \nu^{1/2-\beta}a(\delta, 2k) \\ &= \nu^\beta a(\delta, 2k+1) + \nu^{-\beta}a(\delta, 2k+1).\end{aligned}$$

Let $\pi \in \mathrm{Irr}$ be hermitian. Then $\pi = L((\gamma_1, \dots, \gamma_s))$ for some $\gamma_i \in D$. Now

$$L((\gamma_1, \dots, \gamma_s))^+ = L((\gamma_1^+, \dots, \gamma_s^+))$$

(see §6). This implies that we can write π in the form

$$\pi = L\left(\sum_{i=1}^{n_1}(\nu^{-k_i}\delta_i, \nu^{k_i}\delta_i) + \sum_{i=n_1+1}^{n_2}(\nu^{-k_i-\beta_i}\delta_i, \nu^{k_i+\beta_i}\delta_i) + \sum_{i=n_2+1}^{n_3}(\delta_i)\right)$$

where $\delta_i \in D^u$, $k_i \in (1/2)\mathbb{Z}_+$, $0 < \beta_i < 1/2$ and all $k_1, \dots, k_{n_1} > 0$. Here $n_1, n_2, n_3 \in \mathbb{Z}_+$ and $n_2 \geq n_1$, $n_3 \geq n_2$ (i.e., not all three sums need to show up in the above formula). The two relations at the beginning of the proof and (U4) imply that both

$$\pi \times \left[\prod_{i=1}^{n_1} u(\delta_i, 2k_i - 1)\right] \times \left[\prod_{i=n_1+1}^{n_2} (\nu^{\beta_i - 1/2} u(\delta_i, 2k_i) \times \nu^{1/2-\beta_i} u(\delta_i, 2k_i))\right]$$

and

$$\left[\prod_{i=1}^{n_1} u(\delta_i, 2k_i + 1)\right] \times \left[\prod_{i=n_1+1}^{n_2} (\nu^{\beta_i} u(\delta_i, 2k_i + 1) \times \nu^{-\beta_i} u(\delta_i, 2k_i + 1))\right] \times \left[\prod_{i=n_2+1}^{n_3} u(\delta_i, 1)\right]$$

have

$$L\left(\sum_{i=1}^{n_1} a\,(\delta_i, 2k_i + 1) + \sum_{i=n_1+1}^{n_2} (\nu^{\beta_i} a(\delta_i, 2k_i + 1) + \nu^{-\beta_i} a(\delta_i, 2k_i + 1)) + \sum_{i=n_2+1}^{n_3} (\delta_i)\right)$$

as a composition factor. This completes the proof of the lemma. □

End of proof of Proposition 8.1. Let $\pi \in \mathrm{Irr}^u$. Then, by the preceding lemma there exist σ_i, $\tau_j \in B$ so that $\pi \times \sigma_1 \times \cdots \times \sigma_n$ and $\tau_1 \times \cdots \times \tau_m$ have a composition factor in common. Since (U0), (U1), and (U2) imply that both sides are irreducible, we have

$$\pi \times \sigma_1 \times \cdots \times \sigma_n = \tau_1 \times \cdots \times \tau_m.$$

Since R is factorial and the $u(\delta, n)$'s are prime by (U3), π is a product of some $u(\delta, k)$'s, $\delta \in D$. But the fact that π is hermitian implies that π is actually a product of elements of B. So, we have proved (ii). □

8.3. *Remark.* For $\mathrm{GL}(n)$ over a central simple division F-algebra, we expect that a scheme of this type should work too. In that case a slight modification in the definitions of the $u(\delta, n)$'s and lengths of complementary series is necessary (see [Td7]).

9. On proofs

We have seen that the fulfillment of (U0)–(U4) implies a complete solution of the unitarizability problem for $\mathrm{GL}(n, F)$, as was described in Proposition 8.1.

Let us remark that (U0)–(U4) were expected to hold for $F = \mathbb{C}$ (except maybe (U3) because such questions were not considered). Statements (U1) and (U2) were known, (U0) was expected even by Gelfand and Naimark, while (U4) is easy to prove. A simple consequence of (U0)–(U4), namely, the description of the unitary dual of $\mathrm{GL}(n, \mathbb{C})$, was not generally expected to hold. Now we shall make a few remarks on the history and proofs of (U0)–(U4).

We start with the statement (U4), which belongs to the theory of the nonunitary dual. This fact was proved by Zelevinsky in the case of nonarchimedean F for his classification of $\mathrm{GL}(n, F)$. His proof uses induction on Gelfand-Kazhdan derivatives [Ze, Proposition 8.4]. Rodier noticed in [Ro] that Zelevinsky's proof implies (U4) for the Langlands classification in the nonarchimedean case. We proved (U4) in a simple manner for the archimedean case [Td2, Proposition 3.5. and 5.6]. Such a proof is outlined for nonarchimedean F in Remark A.12(iii) of [Td3]. Sometimes one can conclude the equality in (U4), using the Zelevinsky's proof of Proposition 8.5 in [Ze].

9.1. **Proposition.** *Let* $a_i = (\delta_1^i, \dots, \delta_{n_i}^i) \in M(D)$, $i = 1, 2$. *If* $\delta_k^1 \times \delta_m^2 \in \mathrm{Irr}$ *for all* $1 \le k \le n_1$ *and* $1 \le m \le n_2$, *then*

$$L(a_1) \times L(a_2) = L(a_1 + a_2).$$

The above proposition may be helpful in constructing complementary series.

Let us now consider (U1). Certainly if $F = \mathbb{C}$, then there is nothing to prove since as we already mentioned, $D^u = (\mathbb{C}^\times)\hat{}$ and for $\chi \in D^u$, $u(\chi, n) = \chi \circ \det_n$.

For $F = \mathbb{R}$, $D^u \subseteq \mathrm{GL}(1, \mathbb{R})\hat{} \cup \mathrm{GL}(2, \mathbb{R})\hat{}$. Again, (U1) is evident if $\chi \in (\mathbb{R}^\times)\hat{}$. Speh considered the remaining case of $u(\delta, n)$, $\delta \in \mathrm{GL}(2, \mathbb{R})\hat{} \cap D^u$ [Sp2]. She proved unitarity using adelic methods. Surprisingly, it seems that Gelfand and Graev were already aware of this series of representations in the 1950s (see [Sp2, Remark 1.2.2.] about [GfGr]).

For nonarchimedean F, we have determined in [Td3] the representations $u(\delta, n)$ through the ideas presented in §7. Unitarizability is proved there essentially along those lines. It is also possible to prove unitarizability by the method of Speh as was done in the appendix of [Td3]. Note that here $D^u \cap \mathrm{GL}(n, F)\hat{} \neq \varnothing$ for all $n \ge 1$.

For $F = \mathbb{C}$, (U2) was proved by Stein in [St]. In general, (U2) follows from (U0), using the irreducibility of representations $\nu^\alpha u(\delta, n) \times \nu^{-\alpha} u(\delta, n)$, $0 < \alpha < 1/2$, obtained from Proposition 9.1 and from the analytic properties of intertwining operators. There is also another method in the nonarchimedean case presented in [Bn2].

To prove (U3), one considers the $u(\delta, n)$'s as polynomials and proves the irreducibility of these polynomials. Here one uses the fact that R is a graded ring and that $u(\delta, n)$'s are homogeneous elements. In the proof, one uses basic facts about the composition series of generalized principal series representations (one does not need more detailed information, such as that obtained from Kazhdan-Lusztig type multiplicity formulas). It is a bit surprising that, although we do not know how to write down the polynomials $u(\delta, n)$, we can nevertheless carry out the proof. For proofs of (U3) see [Td3] and [Td2]. The statement (U3) is obvious for $u(\delta, 1)$ by Proposition 6.1.

Finally, let us return to (U0). Let P_n denote the subgroup of $\mathrm{GL}(n, F)$ of all matrices with bottom row equal to $(0, \dots, 0, 1)$. Already Gelfand and Naimark noticed the importance of the statement

(I) If $\pi \in \mathrm{GL}(n, F)\hat{}$, then $\pi | P_n$ is irreducible $(n \in \mathbb{N})$.

Actually, they proved the above statement for $F = \mathbb{C}$, for the representations that they expected to exhaust the unitary dual of $\mathrm{GL}(n, \mathbb{C})$. Several people were aware that the above statement implies (U0). For a written proof see [Sh]. Proof of the implication is based on Mackey theory and Gelfand-Naimark models.

Kirillov stated (I) as a theorem in [Ki1] for F an archimedean field. There he sketched a proof. Vahutinskii's classification of representations of $\mathrm{GL}(3, \mathbb{R})\hat{}$ was based on the proof. Having in mind a correspondence obtained by Mackey theory

$$\begin{aligned}\widehat{P}_n \to \mathrm{GL}(n-1, F)\hat{} \cup \mathrm{GL}(n-2, F)\hat{} \cup \cdots \\ \cup \mathrm{GL}(2, F)\hat{} \cup \mathrm{GL}(1, F)\hat{} \cup \mathrm{GL}(0, F)\hat{},\end{aligned}$$

the statement (I) would imply that one would have simpler realizations for representations from $\mathrm{GL}(n, F)\hat{}$. It is from this setup that the name Kirillov model appears. In [Ki1] Kirillov's intention was to prove that $\pi|P_n$ is operator irreducible (i.e., the commutator consists only of scalars), which is enough by Schur's lemma to see the irreducibility of $\pi|P_n$. One takes any T from the commutator of $\pi|P_n$ and considers a distribution

$$\Lambda_T : \varphi \mapsto \mathrm{Trace}\,(T\pi(\varphi))$$

on $\mathrm{GL}(n, F)$ which is invariant for conjugations with elements from P_n, since T is P_n-intertwining. Now if Λ_T is $\mathrm{GL}(n, F)$-invariant, then using the irreducibility of π, it is not difficult to obtain that T must be a scalar. Kirillov indicated that he proved in [Ki1] that Λ_T is $\mathrm{GL}(n, F)$-invariant (however, see below). This property of the distribution is especially easy to see when π is a continuous finite-dimensional representation. Then the distribution Λ_T is given by a continuous function which must be constant on P_n-conjugacy classes (Λ_T is P_n-invariant). Since in $\mathrm{GL}(n, F)$, $\mathrm{GL}(n, F)$-conjugacy classes contain dense P_n-conjugacy classes, Λ_T must be $\mathrm{GL}(n, F)$-invariant.

Bernstein proved in [Bn2] that for F nonarchimedean, each P_n-invariant distribution is $\mathrm{GL}(n, F)$-invariant. He proved (I) using essentially the Kirillov's strategy. Besides proving (U0) in the nonarchimedean case, he gave a different proof of the implication (I) $\Rightarrow$ (U0).

Bernstein states in [Bn2] that Kirillov's proof of (I) for F archimedean in [Ki1] is incorrect, and he wrote that he himself had an almost complete proof (see [Bn2, p. 55]). In any case, there is no written complete proof of (I) in the archimedean case now. We would say that Kirillov's proof is incomplete rather than incorrect. He failed to give a complete argument that the distribution Λ_T is $\mathrm{GL}(n, F)$-invariant. As was noted by Bernstein, the tools used in Kirillov's paper do not seem to be sufficient for proving (I). The distribution Λ_T is a very special one. Actually (I) would imply that it is a multiple of an irreducible character; so by Harish-Chandra's regularity theorem, it is locally L^1 and analytic on regular semisimple elements. So if one proves that the (eigen) distribution Λ_T is locally L^1 and analytic on regular semisimple elements, one could apply finite-dimensional argument. This may provide a strategy to prove (I) in the archimedean case. This would be a longer proof, and there are also some disadvantages in proving (U0) through (I). We shall say a few words about these disadvantages. Before that, observe that there is an implicit proof of (U0) in [Vo3].

We have mentioned that the approach to the unitary dual of $\mathrm{GL}(n, F)$ is expected to be applicable to $\mathrm{GL}(n)$ over a central division F-algebra A. As far as we understand, Vogan's description of the unitary dual of $\mathrm{GL}(n, \mathbb{H})$ confirms this. Here (U0) cannot be proved through (I), simply because (I) is false in general in this case. The simplest example can be obtained for $\mathrm{GL}(2, A)$ $(A \neq F)$. It is not difficult to see that there exist (irreducible) tempered representations which are reducible when restricted to a nontrivial parabolic subgroup. Thus, it may be more reasonable to search for proof of (U0) which works also for division algebras. There are some candidates for it (see the last remark at the end of this section).

After all, we have the following:

Theorem. *Let F be a locally compact nondiscrete field. Set*

$$B = \{u(\delta, n),\ \ [\nu^{\alpha}u(\delta, n)] \times [\nu^{-\alpha}u(\delta, n)],\ \ \delta \in D^u,\ \ n \in \mathbb{N},\ \ 0 < \alpha < \tfrac{1}{2}\}.$$

Then

(i) *if $\sigma_1, \ldots, \sigma_k \in B$, then $\sigma_1 \times \cdots \times \sigma_k \in \mathrm{Irr}^u$;*
(ii) *if $\pi \in \mathrm{Irr}^u$, then there exist $\sigma_1, \ldots, \sigma_m \in B$, unique up to a permutation, such that*

$$\pi = \sigma_1 \times \cdots \times \sigma_m.$$

We remind the reader once more that there is no written complete proof yet of (U0) in the archimedean case, but there is a complete proof [Vo3] of the above theorem in this case.

Remarks. (1) We give in [Td4] a concrete realization of the topological space $\mathrm{GL}(n, F)\hat{}$ when F is nonarchimedean.

(2) The last theorem, together with [Ka], implies that $\pi \in \mathrm{SL}(n, \mathbb{C})\hat{}$ is isolated if and only if π is the trivial representation and $n \neq 2$.

(3) Let F be nonarchimedean. Let $\rho \in \mathrm{Irr}^u$ be a representation having a nontrivial compactly supported modulo center matrix coefficient. Then for $k \in \mathbb{N}$, the representation

$$\nu^{(k-1)}\rho \times \nu^{(k-3)}\rho \times \cdots \times \nu^{-(k-1)}\rho$$

has a unique square integrable subquotient which will be denoted by $\delta(\rho, k)$. In this way one obtains all D^u (see [Ze] and [Jc1]). In [Td4] we have proved that $\pi \in \mathrm{GL}(n, F)\hat{}$ is isolated modulo center if and only if π equals some $u(\delta(\rho, k), m)$ with $k \neq 2$ and $m \neq 2$.

(4) One could try to prove (U0) for $\mathrm{GL}(n)$ over local division algebras by proving first the following conjecture: Let A be a central local simple algebra, let S be the subgroup of the diagonal matrices in $\mathrm{GL}(2, A)$, let N be the subgroup of upper triangular unipotent elements in $\mathrm{GL}(2, A)$, and let σ be an irreducible unitary representation of S. Then $\mathrm{Ind}_{SN}^{\mathrm{GL}(2, A)}(\sigma)$ should be irreducible.

(5) A proof of (U3) and (U4) for $\mathrm{GL}(n)$ over a local nonarchimedean division algebra is contained in [Td7].

REFERENCES

The references that we include here are directed more to an inexperienced reader in the field of the representation theory than to the experienced one. This is the reason that we have classified them into several groups (see below). We have tried to avoid too many very technical references, which are very common in the field. We include a number of expository and survey papers (they also omit many technicalities). A more demanding reader can also find a choice of relevant references for further reading. We have omitted a number of important references in order not to confuse a reader who is not very familiar with the field.

Let us explain our classification of the references into several groups. The classification is not very rigid. To each reference we have attached a superscript that indicates the group where it belongs. Here is a description of the groups.

sur denotes the group of survey and expository papers.

gen denotes the group of general references.

GL (n) denotes the group of papers that are directly related to the topic of our paper. They are very useful for the further understanding of the topics discussed in our paper. The complete proofs that are omitted in this paper can be found in the papers from this group.

his denotes the group of the historically important references for the topic of this paper.

Lan denotes the group of references directly related to the Langlands program and groups $\mathrm{GL}(n)$. These references are very often related to the various correspondences predicted by the Langlands program for groups $\mathrm{GL}(n)$. Generally, these references are directed to more demanding and experienced readers.

oth denotes the group of remaining useful references. They are mostly related to unitarity.

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