

## THE GENUS-MINIMIZING PROPERTY OF ALGEBRAIC CURVES

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**ABSTRACT.** A viable and still unproved conjecture states that, if  $X$  is a smooth algebraic surface and  $C$  is a smooth algebraic curve in  $X$ , then  $C$  realizes the smallest possible genus amongst all smoothly embedded 2-manifolds in its homology class. A proof is announced here for this conjecture, for a large class of surfaces  $X$ , under the assumption that the normal bundle of  $C$  has positive degree.

### 1. INTRODUCTION

If  $X$  is a smooth 4-manifold and  $\xi$  is a 2-dimensional homology class in  $X$ , one can always represent  $\xi$  geometrically by an oriented 2-dimensional surface  $\Sigma$ , smoothly embedded in the 4-manifold. Depending on  $X$  and  $\xi$  however, the genus of  $\Sigma$  may have to be quite large: it is not always possible to represent  $\xi$  by an embedded sphere. It is natural to ask for a representative whose genus is as small as possible, or at least to enquire what the genus of such a minimal representative would be. Although not much is known about this question in general, there is an attractive conjecture concerning the case that  $X$  is the manifold underlying a smooth complex-algebraic surface. The conjecture is best known in the case that  $X$  is the complex projective plane  $\mathbb{C}P^2$ , in which case it is often attributed to Thom, but the statement seems still to be viable more generally [1].

**Conjecture 1.** *Let  $X$  be a smooth algebraic surface and  $\xi$  a homology class carried by a smooth algebraic curve  $C$  in  $X$ . Then  $C$  realizes the smallest possible genus amongst all smoothly embedded 2-manifolds representing  $\xi$ .*

The attractiveness of this conjecture stems from the connection to which it points, between low-dimensional topology and complex geometry. Through the work of Donaldson in particular, this connection is now a familiar feature of differential topology in dimension 4, and the techniques of gauge theory provide a natural starting point for an approach to the problem. The conjecture was proved in [5] for the special case that  $X$  is a  $K3$  surface, and the results of that paper also gave a lower bound for the genus of an embedded surface in more general complex surfaces. However, in all applicable cases other than  $K3$ , the lower bound proved in [5] falls short of Conjecture 1. The purpose of this present paper is to describe a result which establishes the correctness of the conjecture for a large class of complex surfaces. The hypotheses of Theorem 2 still exclude the tantalizing case of  $\mathbb{C}P^2$ , but conditions (a) and (c) of the theorem admit very many (and conjecturally all) simply connected surfaces  $X$  of

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general type and odd geometric genus. Full details of the proof will appear later in [4].

**Theorem 2.** *The above conjecture holds at least under the following assumptions concerning  $X$  and  $C$ :*

- (a) *the surface  $X$  is simply connected;*
- (b) *the self-intersection number  $C \cdot C$  is positive;*
- (c) *there is a class  $\omega \in H_2(X, \mathbb{C})$  dual to a holomorphic 2-form on  $X$ , such that  $q_k(\omega + \bar{\omega}) > 0$  for sufficiently large  $k$ , where  $q_k$  denotes Donaldson's polynomial invariant.*

Some comments are needed concerning the third hypothesis. Donaldson's polynomial invariants [2] are homogeneous polynomial functions  $q_k$  on  $H_2(X)$ , defined using instanton moduli spaces for structure group  $SU(2)$ ; their degree depends on the parameter  $k$  as well as the homotopy type of  $X$ . Condition (c) appears in [8], where it is shown that this condition will hold for a surface  $X$  of general type provided that:

- (i) the geometric genus  $p_g(X)$  is odd; and
- (ii) the canonical linear system of the minimal model of  $X$  contains a smooth curve.

(This result also rests on some more technical material in [6].) The first of these two conditions ensures that the polynomial invariants have even degree and certainly is essential as long as one considers the invariants associated with the structure group  $SU(2)$ , though the  $SO(3)$  moduli spaces can be used to treat some of the remaining cases. The importance of condition (ii) is less clear, but it does indicate that one should expect (c) to be a rather general property of complex surfaces whenever the polynomial invariants have even degree.

The basic material of the proof of Theorem 2 is the same as that of the main theorem of [5], namely, the moduli spaces of instantons on  $X$  having a singularity along an embedded surface. The structure of the argument, however, is rather different. The difficulty of embedding 2-dimensional surfaces in four dimensions stems from one's inability to remove unwanted self-intersection points of an immersed surface, even when these intersections cancel algebraically in plus-minus pairs; this is the failure of the Whitney lemma in dimension 4 and lies at the heart of 4-manifold topology and all its problems. The first stage of the proof of Theorem 2 is the construction of invariants which measure an obstruction to the removal of such pairs of intersection points. Given an immersed surface  $\Sigma$  with normal crossings in a 4-manifold  $X$ , we use the moduli spaces of singular instantons to define an invariant of the pair  $(X, \Sigma)$ ; this will be a distinguished function  $R_d(s) : H_2(X) \rightarrow \mathbb{R}$ , taking the form of a homogeneous polynomial of degree  $d$  on  $H_2(X)$  and a finite Laurent series in the formal variable  $s$ . This invariant has the property that the order of vanishing of  $R_d$  at the point  $s = 1$  gives an upper bound on the number of positive-signed intersection points in  $\Sigma$  which can ever be removed by a homotopy of the immersion. The second stage of the proof is to show that, in the case of an algebraic curve in a suitable algebraic surface, the invariants  $R_d(s)$  give information which is sharp enough to establish the assertion of Conjecture 1; this entails proving a nonvanishing theorem for the value of  $R_d(s)$  at  $s = 1$ . Using ideas from [8], we shall in fact show that, if  $C'$  is an irreducible algebraic

curve with a single ordinary double point in a suitable complex surface  $X$ , then  $R_d(1)$  is positive for the pair  $(X, C')$  when evaluated on a class  $\omega + \bar{\omega}$  as in (c), so showing that the self-intersection point in  $C'$  cannot be removed by any homotopy. Theorem 2 is easily deduced from this.

The structure of this proof is closely modeled on Donaldson's proof of the indecomposability of complex surfaces in [2], but a closer parallel still is in [1], where a similar strategy was used in connection with Conjecture 1. In that paper, use was made of the instanton moduli spaces associated with a branched double cover  $\tilde{X} \rightarrow X$ , branched along  $\Sigma$ . The moduli spaces of singular instantons which we use here can, in some circumstances, be interpreted as moduli spaces of suitably *equivariant* instanton connections on such a covering manifold (equivariant, that is, under the covering involution). The replacing of the full moduli space on  $\tilde{X}$  by this equivariant part can be seen as the main difference in the framework of the argument between [1] and the present paper. The other main new ingredient here is the organization of the invariants obtained from the singular instantons to form the finite Laurent series  $R_d$ . The remaining two sections of this paper provide some further details of the proof; some of this material is rather technical, particularly in §3. It seems likely that a change of strategy in this last part of the argument will eventually lead to a slightly more general result than Theorem 2.

## 2. OBSTRUCTIONS TO REMOVING INTERSECTION POINTS

Let  $X$  be a smooth, oriented, simply connected closed 4-manifold and  $\Sigma$  an embedded (rather than just immersed) orientable surface in  $X$ . Given a Riemannian metric on  $X$ , it was shown in [5] how one can construct moduli spaces  $M_{k,l}^\alpha$  associated to the pair; roughly speaking,  $M_{k,l}^\alpha$  parametrizes the finite-action anti-self-dual  $SU(2)$  connections on  $X \setminus \Sigma$  with the property that, near to  $\Sigma$ , the holonomy around small loops linking the surface is asymptotically  $\exp 2\pi i \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix}$ . Here  $\alpha$  is a real parameter in the interval  $(0, 1/2)$  and  $k$  and  $l$  are the integer topological invariants of such connections: the "instanton" and "monopole" numbers. For a generic choice of Riemannian metric and away from the flat or reducible connections in the moduli space,  $M_{k,l}^\alpha$  is a smooth manifold of dimension

$$(1) \quad 8k + 4l - 3(b^+ + 1) - (2g - 2)$$

where  $b^+$  is the dimension of a maximal positive subspace for the intersection form on  $H^2(X)$  and  $g$  is the genus of  $\Sigma$ . In the case that  $b^+$  is odd, the dimension is even, and we write it as  $2d(k, l)$ , so that  $d = d(k, l)$  is half of (1). Following Donaldson's definition [2] of the polynomial invariants  $q_k$ , it was shown in [5] that the moduli spaces  $M_{k,l}^\alpha$  can be used, when  $b^+$  is odd, to define a homogeneous polynomial function of degree  $d(k, l)$ ,

$$q_{k,l} : H_2(X) \rightarrow \mathbb{R}.$$

(In [5], this polynomial was defined only on the orthogonal complement of  $[\Sigma]$  in  $H_2(X)$ , and this is all we will actually need; the definition, however, can be extended to the whole homology group. Also, a 'homology orientation' of  $X$  is needed to fix the overall sign.) When  $b^+$  is at least three, the polynomial  $q_{k,l}$  is independent of the parameter  $\alpha$  and the Riemannian metric; it is an invariant of the pair  $(X, \Sigma)$ .

Because of the way  $k$  and  $l$  enter the dimension formula (1), the degree of  $q_{k-1, l+2}$  is the same as the degree of  $q_{k, l}$ . It is natural to combine all the polynomials of a given degree into one Laurent series:

$$R_d(s) = \sum_{d(k, l)=d} s^l q_{k, l}.$$

This series is actually finite in both directions, though this is not obvious a priori from its definition. Note that, depending on the parity of the genus  $g$  and the value of  $b^+ \bmod 4$ , the invariant  $R_d$  will be defined only for  $d$  of one particular parity. Flat connections on the complement of  $\Sigma$  can cause difficulties in defining the invariants directly from the moduli spaces, but these can be overcome, for example, by a device such as that described in [7].

Having defined invariants for embedded surfaces, we can now define invariants for immersed surfaces with normal crossings. It would seem feasible to do this by directly using gauge theory on the complement of the immersed surface, but a short-cut is available. We shall convert such an immersed surface  $\Sigma$  into an embedded surface by blowing up  $X$  at the intersection points. This is the process modeled on the situation in complex geometry, where a curve  $\Sigma$  with a normal crossing at  $p$  is replaced by its proper transform, which is a smooth curve in the new surface  $\tilde{X} = X \# \overline{\mathbb{C}\mathbb{P}^2}$ , the blow-up of  $X$  at  $p$ . In the  $C^\infty$  case, the model is just the same; there are really two different cases according to the sign of the intersection point, but no essential difference in the local picture. Thus we obtain an embedded surface  $\tilde{\Sigma}$  in a new manifold  $\tilde{X} = X \# n \overline{\mathbb{C}\mathbb{P}^2}$ . We define the invariants  $q_{k, l}$  and  $R_d$  for  $(X, \Sigma)$  to be the restriction of the invariants of  $(\tilde{X}, \tilde{\Sigma})$  to  $H_2(X) \subset H_2(\tilde{X})$ .

The next stage in the argument is to see how the invariant  $R_d(s)$  changes when the immersion of  $\Sigma$  in  $X$  is changed by a homotopy. During a homotopy, double points can appear and disappear in  $\Sigma$  in quite complicated ways, but standard theory says that after a small perturbation any such changes can be broken down into a combination of moves, each of which is one of six types. One has to consider the following three standard modifications and their inverses (see [3] for pictures and explanations of these):

- (a) introduce a positive double point by a twist move;
- (b) introduce a negative double point by a twist move;
- (c) introduce a cancelling plus-minus pair by a finger move.

(We should emphasize that we are talking about homotopies whose starting and finishing points are immersions; only (c) can be achieved by a homotopy *through* immersions.) The change in  $R_d$  under each of these three moves is summarized by the following proposition.

**Proposition 3.** *Let  $\Sigma$  be obtained from  $\hat{\Sigma}$  by one of the moves just described, and let  $R_d$  and  $\hat{R}_d$  be the invariants for  $(X, \Sigma)$  and  $(X, \hat{\Sigma})$ . Then, according to the three cases, we have:*

- (a)  $R_d(s) = (1 - s^{-2})\hat{R}_d(s)$ ;
- (b)  $R_d(s) = \hat{R}_d(s)$ ;
- (c)  $R_d(s) = (1 - s^{-2})\hat{R}_d(s)$ .

As a consequence of these relations, the order of vanishing of  $R_d(s)$  at  $s = 1$  increases by one every time a positive double point is introduced by either of

the moves (a) or (c) and decreases by one every time a positive double point disappears. So, as was stated in the introduction, the order of vanishing of  $R_d$  at  $s = 1$  puts an upper bound on the number of positive double points which can be removed.

Note also that the invariants  $R_d$  are unable to detect subtleties of knotting: two embedded surfaces of the same genus will have the same invariant if the embeddings are homotopic. (There is also a simple formula for how  $R_d$  changes when the genus of  $\Sigma$  is increased by summing with a torus in  $X$ , but this involves aspects which would take us too far afield; see [4].)

The proof of Proposition 3 involves some rather simple gluing arguments. Consider (a) for example. After introducing the positive double point to form  $\Sigma$  from  $\widehat{\Sigma}$ , the definition of the invariants for immersed surfaces tells us to remove the double point by blowing up, to get  $(\widetilde{X}, \widetilde{\Sigma})$ . Examining the overall effect, we find that, up to diffeomorphism,  $(\widetilde{X}, \widetilde{\Sigma})$  is the connect sum of the pair  $(X, \widehat{\Sigma})$  with the pair  $(\overline{\mathbb{C}\mathbb{P}^2}, \overline{C})$ , where  $C$  is a conic curve in the projective plane. One must analyse this connected sum of pairs to show that the invariants for  $\widetilde{\Sigma}$  and  $\widehat{\Sigma}$  are related by  $q_{k,l} = \widehat{q}_{k,l} - \widehat{q}_{k-1,l+2}$ . (The sign here is rather subtle and crucial to the argument.) Technical aspects of the gluing construction, as well as some aspects of the algebraic geometry in §3, can be considerably simplified by using the fact that the moduli spaces  $M_{k,l}^\alpha$  for  $\alpha = 1/4$  are essentially equivalent to moduli spaces of equivariant connections on a branched double cover, or to orbifold connections in the case that a global double covering does not exist. So, in the example above, rather than think of forming the connect sum along a pair  $(S^3, S^1)$ , one may think instead of gluing across a copy of  $S^3$ , with invariance imposed under an involution.

### 3. THE COMPLEX CASE

Suppose now that  $X_1$  is a smooth complex surface and  $C_1$  is a smooth algebraic curve in  $X_1$ . If the self-intersection number of  $C_1$  is positive, and we wish to prove Conjecture 1 for the homology class  $[C_1]$ , then it turns out to be enough to tackle instead the homology class  $n[C_1]$  for any large  $n$  (see [1] or [5] for this elementary construction). So let  $C_2$  be a smooth curve in the linear system  $|nC_1|$ . Taking  $n$  large enough, we may suppose that the linear system  $|C_2|$  contains, in addition to this smooth curve, an irreducible singular curve  $C'_2$  with a single ordinary double point. We can look at  $C'_2$  as an immersed 2-manifold with a single normal crossing of positive sign; its genus is one less than the genus of  $C_2$ . Suppose the conjecture fails for the homology class of  $[C_2]$ . Then we can find an *embedded* surface  $\Sigma$  in the this class, with the same genus as the immersed surface  $C'_2$ . Since  $X_1$  is simply connected,  $\Sigma$  and  $C'_2$  will be homotopic, and it follows from the results of the previous section that the invariant  $R_d(s)$  for  $(X_1, C'_2)$  vanishes at  $s = 1$ . To obtain a contradiction and prove the conjecture, we therefore need a nonvanishing theorem which states that  $R_d(1)$  is nonzero. We shall in fact prove that, if the genus of  $C'_2$  is odd and its homology class is even (conditions which eventually entail no loss of generality) and if the conditions of Theorem 2 hold, then the value of  $R_d(1)$  on the class  $\omega + \overline{\omega}$  of (c) is positive once  $d$  is sufficiently large. If we recall again that the invariants for an immersed surface are defined in terms of the embedded surface obtained by blowing up, we are led to blow up  $(X_1, C'_2)$  at

the single double point of  $C'_2$  to obtain finally a smooth algebraic pair  $(X, C)$ . The following result is therefore what is wanted.

**Proposition 4.** *Let  $C$  be a smooth curve in an algebraic surface  $X$ , satisfying the conditions of Theorem 2. Suppose in addition that  $C$  has odd genus and that its homology class has divisibility 2 in  $H_2(X, \mathbb{Z})$ . Then for large  $d$  the value of the invariant  $R_d(1)(\omega + \bar{\omega})$  for the pair  $(X, C)$  is strictly positive.*

Note that since  $R_d(1)$  is just the sum of the invariants  $q_{k,l}$  of a given degree, it will suffice to show that each of these terms is nonnegative and that at least one of them is positive. Under some conditions on  $\alpha$  and the metric on  $X$ , it was shown in [5] that the moduli spaces  $M_{k,l}^\alpha$  can be interpreted as moduli spaces of stable parabolic bundles. It is therefore tempting to try to adapt Donaldson's argument in [2] to prove that each  $q_{k,l}$  is positive when evaluated on the hyperplane class, provided that the degree  $d$  is large. (Actually this would not be quite the right thing; one should construct a different version of  $q_{k,l}$  which varies with  $\alpha$ , to take account of how the natural polarization of a moduli space of parabolic bundles changes as the parabolic weight is varied.) Unfortunately, there is an obstruction to this programme. A key technical step in the argument of [2] is to show that, once  $d$  is large, the moduli spaces of stable bundles on a complex surface have the dimension one would naively predict from the index formula, namely,  $d$ . This "regularity" result is false in the context of parabolic bundles.

To explain what is true in the way of regularity, it is convenient to introduce the "magnetic charges"  $m_1 = k$  and  $m_2 = k + l - (\Sigma \cdot \Sigma)/4$  in place of  $k$  and  $l$ . The naively expected complex dimension of the moduli space  $M_{k,l}^\alpha$  is then  $d \sim 2m_1 + 2m_2$ , according to the formula (1). In order for the moduli space to have this expected dimension, it is not enough that  $d$  alone be large—it is necessary for both  $m_1$  and  $m_2$  to be large.

Fortunately, a general vanishing theorem was proved in [5] which shows that, if the absolute value of the difference  $|m_1 - m_2|$  is larger than a quantity  $(K_X \cdot C)/4$ , then the invariants  $q_{k,l}$  of a pair  $(X, C)$  vanish when restricted to homology classes orthogonal to  $[C]$  in  $H_2(X)$ . So, if we take such a homology class, then the moduli spaces which might have the wrong dimension (where only one of  $m_1$  or  $m_2$  is large) will not contribute. Taking this line forces us to abandon the idea of following Donaldson's argument from [2], since the hyperplane class is not orthogonal to  $C$ . Instead, we adapt O'Grady's argument from [8].

As was mentioned in the introduction, the results of [8] show that, under suitable conditions on a complex surface  $X$  and  $k$ , the value  $q_k(\omega + \bar{\omega})$  of the ordinary polynomial invariant is strictly positive when  $\omega$  is dual to a generic holomorphic 2-form. (Note that such a class is always going to be orthogonal to a holomorphic curve such as  $C$ .) Part of the argument adapts readily to the parabolic case to show that  $q_{k,l}(\omega + \bar{\omega})$  is at least nonnegative, provided only that the moduli spaces have the correct regularity properties. All that remains finally is to show that at least one of these values is nonzero. The last ingredient is another hard result from [5] which shows that, for the special value of the monopole number  $l = (g - 1)/2$ , the invariant  $q_{k,l}$  for the pair  $(X, C)$  is equal to  $2^g q_k$  when restricted to the orthogonal complement of  $C$ . So, for this special value of  $l$ , the nonvanishing of  $q_{k,l}$  can be deduced from the nonvanishing of  $q_k$ .

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