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GENERA OF ALGEBRAIC VARIETIES AND COUNTING OF LATTICE POINTS

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ABSTRACT. This paper announces results on the behavior of some important algebraic and topological invariants — Euler characteristic, arithmetic genus, and their intersection homology analogues; the signature, etc. — and their associated characteristic classes, under morphisms of projective algebraic varieties. The formulas obtained relate global invariants to singularities of general complex algebraic (or analytic) maps. These results, new even for complex manifolds, are applied to obtain a version of Grothendieck-Riemann-Roch, a calculation of Todd classes of toric varieties, and an explicit formula for the number of integral points in a polytope in Euclidean space with integral vertices.

Consider first the behavior of the classical Euler-Poincare characteristic

$$e(X) = \sum_i (-1)^i \operatorname{rank} H^i(X)$$

under a (surjective) projective morphism $f\colon X \to Y$ of projective (possibly singular) algebraic varieties. Such a morphism can be stratified with subvarieties as strata. In particular, there is a filtration of Y by closed subvarieties, underlying a Whitney stratification,

$$\phi \subset Y_0 \subset \cdots \subset Y_s = Y$$

of strictly increasing dimension, such that $Y_i - Y_{i-1}$ is a union of smooth manifolds of the same dimension and such that the restriction of f to $f^{-1}(Y_i - Y_{i-1})$ is a locally trivial map of Whitney stratified spaces. (In the results it will suffice to have $\dim Y_i < \dim Y_{i+1}$ and $\dim f^{-1}(x)$ constant over "strata" $Y_i - Y_{i-1}$.)

We recall the definition of the normal cone $C_Z W$ of an irreducible subvariety Z of a variety W:

$$C_Z W = \operatorname{Spec}\left(\bigoplus \mathcal{J}^n/\mathcal{J}^{n+1}\right) ,$$

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$\mathscr{I}$ the sheaf of ideals defining Z. Let $P(C_Z \oplus 1)$ be its projective completion, and let $P_{Z,W}$ be the general fiber of the canonical projective morphism [F1]

$$P(C_Z \oplus 1) \to Z .$$

For example, if Z is a smooth subvariety of W of complex codimension d, $P_{Z,W}$ is the complex projective space CP^d of dimension d.

Let $\mathscr{V}$ be the set of components of strata of Y. For $V \in \mathscr{V}$ define $\hat{e}(\overline{V})$ inductively by the formula

$$\hat{e}(\overline{V}) = e(\overline{V}) - \sum_{W<V} \hat{e}(\overline{W})\, e(P_{W,V}) ,$$

where the sum is over all $W \in \mathscr{V}$ with $W \subset \overline{V} - V$. Let $P_{V,f}$ be the general fiber of the composite

$$P(C_{f^{-1}V} \oplus 1) \longrightarrow f^{-1}V \longrightarrow V .$$

Theorem 1. *Let $f\colon X \to Y$ be a surjective morphism of projective complex algebraic varieties, and let $\mathscr{V}$ be the components of the "strata". Assume that $\pi_1 V = 0$ for $V \in \mathscr{V}$. Let F be the general fiber of f. Then*

$$(*) \qquad e(X) = e(Y)\, e(F) + \sum_{\mathscr{V}_0} \hat{e}(\overline{V}) \left[e(P_{V,f}) - e(F)\, e(P_{V,Y}) \right] ,$$

$\mathscr{V}_0$ the subset of $\mathscr{V}$ with $\dim V < \dim Y$.

Example. Let X be obtained from Y^n by blowing up a point y. Let $D = P(C_{\{y\}}) = f^{-1}y \subset X$ be the exceptional set. Then $(*)$ becomes

$$e(X) = e(Y) + 2e(D) - e(P(C_{\{y\}} \oplus 1) .$$

If y is a smooth point, D and $P(C_{\{y\}} \oplus 1)$ are complex projective spaces CP^{n-1} and CP^n, and this is the well-known result

$$e(X) = e(Y) + n - 1 .$$

Definition. Any invariant satisfying $(*)$ under the hypotheses of Theorem 1 will be said to have the stratified multiplicative property (SMP).

Without the π_1 hypothesis the SMP and the present results must be phrased in terms of coefficients in local systems. Theorem 1 for this case essentially includes as a corollary results like the generalization of Riemann-Hurwitz given in [DK; I; K, (III, 32)].

As another example consider the signature $\sigma(X)$, defined in [GM2] as the signature (number of positives minus number of negatives in a diagonalization) of the intersection form in the middle-dimensional intersection homology $\mathrm{IH}_n^{\overline{m}}(X)$ with middle perversity, X projective of complex dimension n. We show the signature also has the SMP. For blowing up a point this becomes

$$\sigma(X) = \sigma(Y) - \sigma(P(C_{\{y\}} \oplus 1) .$$

In [CS1] a different formula was given for the behavior of the signature (and the L-classes) under any stratified map. In the case of blowing up a point the formula of [CS1] gives

$$\sigma(X) = \sigma(Y) + \sigma(E_y) ,$$

where

$$E_y = f^{-1}(N) / f^{-1}(L) \ ,$$

N a (piecewise linear) neighborhood of $y \in V$ and $L = \partial N$ its frontier. There are two key differences between this formula and the above. The first is that the topological completion E_y of $f^{-1}(\operatorname{Int} N)$ usually is not an algebraic variety. The second is that, unlike E_y, it can and usually does happen that no open neighborhood of $f^{-1}(y)$ in $P_{V,f}$ (which is actually a bundle over D) embeds topologically in X, extending the inclusion of $f^{-1}(y) \subset X$. Thus the behavior of genera under *algebraic* maps can be determined without knowing the local topological structure around strata precisely.

We now consider two extensions of Hirzebruch genera of smooth varieties to general varieties. Let $Ih^{p,q}(X)$ be the Hodge numbers of Saito's pure Hodge decompositon [S] on $\mathrm{IH}_*^{\overline{m}}(X;\mathbb{C})$, and let $h_i^{p,q}(X)$ be the Hodge numbers of Deligne's mixed Hodge structure on $H^i(X;\mathbb{C})$. Let y be a variable, and define genera and intersection genera by

$$\chi_y(X) = \sum_p \left[\sum_{i,q} (-1)^{i-p} h_i^{p,q}(X) \right] y^p$$

and

$$I\chi_y(X) = \sum_p \left[\sum_q (-1)^q Ih^{p,q}(X) \right] y^p \ .$$

Thus $\chi_{-1} = e$, $I\chi_1 = \sigma$, and $I\chi_0$ and χ_0 are two possible extensions to singular varieties of the arithmetic genus.

Theorem 2. *The genera χ_y and $I\chi_y$ have the SMP.*

We discuss briefly the proof of Theorem 2 for the intersection genera. The strategy is to argue inductively, blowing up closures of strata starting with a lowest dimensional component, relying on the following result on general blowing up:

Theorem 3. *Let X be a projective complex algebraic variety and Z a closed subvariety. Let $\widetilde{X} = Bl_Z X$ be the blowup of X along Z, with exceptional divisor $E = P(C_Z X)$. Then*

$$I\chi_y(\widetilde{X}) = I\chi_y(X) + (1-y)\chi_y(E) - I\chi_y(P(C_Z X \oplus 1)) \ .$$

This is proved by considering a diagram:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\psi}} & \widetilde{X} \cup_E P(C_E \widetilde{X} \oplus 1) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\psi} & \widetilde{X} \cup_E P(C_Z X \oplus 1) \end{array}$$

The vertical maps are the projection from the blowups along $Z \subset X$ and along the canonical embedding

$$Z \subset C_Z X \subset P(C_Z X \oplus 1) \subset \widetilde{X} \cup_E P(C_Z X \oplus 1) \ ;$$

note that

$$P(C_E \widetilde{X} \oplus 1) = Bl_Z P(C_Z X \oplus 1) \ .$$

The horizontal arrows are the specialization maps from the general to the special fiber in the deformation to the normal cone [H, V, GM1]. For the top map note that $Bl_E\widetilde{X} = \widetilde{X}$. The main idea is to compare the monodromy weight filtrations and associated spectral sequences for $R\psi_*\mathbb{IC}^\bullet(X)$ and $R\tilde{\psi}_*\mathbb{IC}^\bullet(\widetilde{X})$, taking advantage of the result of [S], strengthening [BBD], that the vertical maps "induce" injections in intersection cohomology onto summands, respecting pure Hodge structures.

For each genus or intersection genus we define corresponding characteristic homology class T_y or IT_y. For example, T_{-1} is the total MacPherson Chern class, $T_0 = Td$ (the image in homology of) the Todd class of [BFM], appearing in the generalized GRR theorem (see also [F1]), and IT_1 is the Goresky-MacPerhson L-class. We show these also have the SMP; i.e., they satisfy the appropriate version of $(*)$. Combining this with Grothendieck-Riemann-Roch yields, for example:

Corollary. *Let $f\colon X \to Y$ be as in Theorem* 1. *Let α be a locally free sheaf on X, and let $f_*\alpha$ be a locally free sheaf on Y with*

$$f_*(\alpha \otimes \mathscr{O}_X) = f_*\alpha \otimes \mathscr{O}_Y \ .$$

(*This always exists for Y smooth.*) *Then*

$$\begin{aligned} &f_*Td_X - \chi(F\,;\mathscr{O}_F)\,ch^{-1}(f_*\alpha) \cap f_*(ch(\alpha) \cap Td_X) \\ &\quad = \sum_{\mathscr{V}_0} i_*\widehat{T}d_{\overline{V}}\left[\chi(P_{V,f}\,;\mathscr{O}_{P_{V,f}}) - \chi(F\,;\mathscr{O}_F)\,\chi(P_{V,Y}\,;\mathscr{O}_{P_{V,Y}})\right] \ . \end{aligned}$$

Finally, we consider toric varieties and counting of lattice points. Toric varieties arise naturally in algebraic and symplectic geometry, representation theory, number theory, and combinatorics. A toric variety X^n is an irreducible normal variety on which the complex torus $(\mathbb{C} - \{0\})^n$ acts with an open orbit [O, D]. Equivalently, it is given locally by systems of monomial equations. If X is projective, the quotient of X by the action of the real torus $(S^1)^n$ can be naturally identified (via the "moment map" if X happens to be a symplectic manifold) with a convex polytope in $\mathbb{R}^n$ whose vertices have integral coordinates, and questions about the polytope can often be translated into questions about the variety and vice versa.

Given the polytope P with vertices lattice points, there is a classical recipe for constructing the corresponding toric variety. This is often done via the fan Σ dual to the polytope; in fact, every toric variety has such a description uniquely in terms of a fan [O, 1.2]. A complete fan is a decomposition of $\mathbb{R}^n$ into cones, each spanned by its "rays", i.e., half lines from the origin through an integral point. The duality between faces of P and cones of Σ is given as follows: For each face E of P let $\mathscr{F}_E$ be the set of codimension one faces containing E, and let

$$\check{\sigma}_E = \operatorname{span}\{n_F \mid F \in \mathscr{F}_E\} \ ,$$

where n_F is a primitive integral point, orthogonal to F, and $n_F \cdot (m-p) > 0$ for $p \in F$ and m a vertex not in F. (For $E = P$, $\check{\sigma}_E = \{0\}$.) For example, CP^n is the toric variety determined by the simplex spanned by the unit orthogonal basis vectors $e_1, \dots, e_n$ and the origin or by the complete fan whose rays are the half lines spanned by $e_1, \dots, e_n$, $-e_1 - e_2 - \cdots - e_n$.

For simplicity we assume our toric variety X is simplicial; i.e., each real n-dimensional cone in Σ is spanned by n rays. The present results can be modified to cover the general case. Such an X has rational singularities and can be described locally as the quotient of $\mathbb{C}^n$ by the linear action of a finite cyclic group. In particular, X is an orbifold and satisfies Poincaré duality over $\mathbb{Q}$. A toric variety is smooth if and only if it is simplicial and the integral points on the rays of each cone span over $\mathbb{Z}$ a summand of the lattice of integral points of $\mathbb{R}^n$.

For X simplicial we give an explicit formula for $Td(X) = T_0(X) = IT_0(X) \in H_*(X;\mathbb{C})$. With Riemann-Roch (see [D] or the exposition [F2]) this leads to a formula for the number of integral points in P. Actually, we will consider

$$T(X) = [X] + 2Td_{2n-2}(X) + \cdots + 2^n Td_0(X) .$$

Mock characteristic classes of X may be defined as follows: For each cone $\sigma \in \Sigma$, let $V(\sigma)$ be the corresponding subvariety of X. If $\sigma = \check{\sigma}_E$, $V(\sigma)$ is just the part of X lying over the face E and is, in fact, the corresponding simplicial toric variety. Now pretend the stable complex tangent bundle is a sum of line bundles with Chern classes $[V(\sigma)] \in H_{2n-2}(X;\mathbb{Q}) \cong H^2(X;\mathbb{Q})$, for $\sigma \in \Sigma^{(1)}$ a ray, and take the appropriate polynomial in these, as in [H, §12] for smooth varieties. For example, for the Todd polynomial this gives the Mock Todd class $TD(X)$ of [P] (thought of in homology). Similarly, the mock L-classes

$$L^{(m)}(X) \in H_*(X;\mathbb{Q})$$

are defined using the L-polynomial. For smooth toric varieties, the mock classes agree with the actual ones. For example, for CP^n this gives the well-known formulas

$$L(CP^n) = (c^2/\tanh c^2)^{n+1}$$

and

$$Td(CP^n) = \{c/(1-e^{-c})\}^{n+1} ,$$

c represented by a hyperplane $CP^{n-1} \subset CP^n$. Our mock T-class will be defined as a sum over all cones σ,

$$T^{(m)}(X) = \sum_\sigma L^{(m)}(V(\sigma)) .$$

Next we define some algebraic numbers associated to a k-dimensional simplicial cone σ, essentially measuring "angles" in which faces meet. Let $N \subset \mathbb{R}^n$ be the set of integral points. By replacing N by the plane containing σ (or dividing by its orthogonal complement), it may be assumed without loss of generality that $n = \dim N = k$. Let $\sigma = \langle n_1, \dots, n_k\rangle$ be spanned by $n_i \in N$, $1 \le i \le k$. Let m_i be the unique primitive elements of N with

$$m_i \cdot n_j = 0 , \qquad i \neq j ,$$

and

$$q_i = m_i \cdot n_i > 0 .$$

Let G_σ be the finite abelian group

$$G_\sigma = N/N' , \qquad N' = \mathbb{Z}m_1 + \cdots + \mathbb{Z}m_k .$$

Set $m(\sigma) =$ order of G_σ. These are the groups which act on affine spaces in the orbifold description of X.

For $g = m + N'$ define $\lambda_j(g)$, also denoted $\lambda_{n_j}(g)$,

$$\lambda_j(g) = \exp \frac{2\pi i}{q_j} m \cdot m_j = \exp 2\pi i \, \gamma_{n_j}(g) .$$

Let

$$G_\sigma^o = \{g \in G_\sigma \mid \lambda_j(g) \neq 1 \text{ for } 1 \leq j \leq k\} .$$

Equivalently, G_σ^o consists of the elements of G_σ of the form $m + N'$ with m in the interior of the cone spanned by $m_1, \dots, m_k$.

Now suppose that σ as above is a cone of the simplicial fan Σ. Let ρ_i be the ray through n_i. Then we define a characteristic class $\mathscr{A}(\sigma) \in H_*(X_\Sigma; \mathbb{Q})$ by

$$\mathscr{A}(\sigma) = \frac{1}{m(\sigma)} \sum_{G_\sigma^o} \prod_1^k \frac{\lambda_i(g) e^{2[V(\rho_i)]} + 1}{\lambda_i(g) e^{2[V(\rho_i)]} - 1} .$$

If G_σ^o is empty, we take $\mathscr{A}(\sigma) = 0$, and we also set $\mathscr{A}(\sigma) = 1$ for $\dim \sigma = 0$. The fact that this is a rational class follows from Galois invariance.

Theorem 4. *Let X_Σ be the toric variety corresponding to the complete simplicial fan Σ. Then*

$$T(X_\Sigma) = \sum_\sigma T^{(m)}(V(\sigma)) \cdot \mathscr{A}(\sigma) .$$

This theorem applied in dimension $2n - 4$ gives precisely Pommersheim's calculation [P] of Td_{2n-4} (denoted Td^2 in [P]). Note that the present methods are quite different from those of [P].

Theorem 4 really is an explicit calculation, because the ring $H_*(X; \mathbb{Q}) \cong H^*(X; \mathbb{Q}) \cong A^*(X) \otimes \mathbb{Q}$ is explicitly known (see [O, pp. 134–135]). Therefore, this result allows us to determine the Hilbert polynomial (also called the Ehrhart polynomial) ℓ_P, a degree n polynomial [E] that counts lattice points in integral dilations of P:

$$\ell_P(\mu) = \#\{N \cap \mu P\} , \qquad \mu \text{ a positive integer.}$$

Here is the result for P a simplex, extending Pick's theorem [Pi] on planar polygons and results of Mordell [M] and Pommersheim [P] in dimension three to all dimensions: notation like $m(E)$, etc., refers to $m(\check{\sigma}_E)$; $\nu(E)$ is the volume of the face E, normalized with respect to the intersection of N with $\dim E$ plane containing E; and $\mathscr{H}_E = \mathscr{F}_\phi - \mathscr{F}_E$.

Theorem 5. *Let Δ be an n-simplex with vertex points in the lattice N. Let a_r be the coefficient of U^r in the power series*

$$\sum_{E \leq \Delta} \frac{1}{m(E)} \left[\sum_{K \leq E} \omega_K \prod_{F \in \mathscr{H}_K} \frac{(\nu(F)U)}{\tanh(\nu(F)U)} \right] \sum_{G_E^o} \prod_{F \in \mathscr{F}_E} \coth\{\pi i \gamma_F^E(g) + \nu(F)U\} ,$$

where

$$\omega_K = m(K) \prod_{F \in \mathscr{F}_K} (\nu(F)U) .$$

Let E be any face of Δ of dimension r, and let

$$b_r = \frac{\nu(E)}{2^{n-r} m(E) \prod_{F \in \mathcal{F}_E} \nu(F)} .$$

Then

$$\ell_\Delta(k) = \sum_{r=0}^{n} a_r b_r k^r .$$

These results and known facts about Todd classes or Ehrhart polynomials imply nontrivial number theoretical results (cf. [R1, R2, P]). For other valuable perspectives and results on Todd classes of toric varieties and lattice points, see the works of Brion [B] and of Morelli [Mo].

Note that in this formula all terms are just functions of the volumes of the codimension one faces, the volume of a single face in each dimension, and the cones at each vertex.

The ingredients of the proof of Theorem 4 are Theorem 2 applied to a resolution of X obtained by interior subdivision of the fan Σ, formulas of [H] relating Todd classes and L-classes of smooth varieties, and formulas of Atiyah-Singer and Hirzebruch-Zagier type for equivariant L-classes related to the orbifold structure.

Added May 17, 1993. The first two coefficients of ℓ_Δ (and the constant term) are classically known (essentially "Pick's Theorem"), and Pommersheim's work mentioned above determines the next coefficient. Khovanskii has just communicated to us that recently but independently of Pommersheim (and of the above results determining all the coefficients of ℓ_Δ) he and Kantor have also found one coefficient.

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