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Uncountably categorical theories, by Boris Zilber. Translations of Mathematical Monographs, vol. 117, American Mathematical Society, Providence, RI, 1993, vi + 122 pp., \$97.00. ISBN 0-8218-4586-1

This book is a translation from the Russian of Boris Zilber's *Doktor Nauk* dissertation. It contains a large and important chunk of Zilber's work, most of it having already appeared in various papers from the late 1970s to mid-1980s. This work has had a profound effect on the development and perspectives of modern model theory.

Let me begin by mentioning two (seemingly unrelated) problems solved in this book—one in combinatorial geometry/permutation group theory, the other in first-order logic. The geometrical question requires some background. A *geometry* here is a set X together with a closure operation cl(-) taking subsets of X to subsets of X and with the properties:

(i) $\mathrm{cl}(\varnothing) = \varnothing$.
(ii) $\mathrm{cl}(\{a\}) = \{a\}$ for each $a \in X$.
(iii) $\mathrm{cl}(Y) = \bigcup\{\mathrm{cl}(Y') : Y' \subseteq Y,\ Y' \text{ finite}\}$ whenever Y is a subset of X.
(iv) For any $Y \subseteq X$, $\mathrm{cl}(\mathrm{cl}(Y)) = \mathrm{cl}(Y)$.
(v) If $Y \subseteq X$, $a, b \in X$, and $b \in \mathrm{cl}(Y \cup \{a\}) \setminus \mathrm{cl}(Y)$, then $a \in \mathrm{cl}(Y \cup \{b\})$.

The geometry (X, cl) is called *locally finite* if the closure of any finite subset of X is finite. The geometry is called *homogeneous* if for any closed subset Y of X and $a, b \in X \setminus Y$ there is an automorphism of the geometry which fixes Y pointwise and takes a to b. Examples of (countably) infinite, homogeneous, locally finite geometries are:

(a) degenerate type: where $\mathrm{cl}(Y) = Y$ for all $Y \subseteq X$;
(b) projective type: X is an infinite-dimensional projective space over a finite field F, and cl(-) is F-linear closure;
(c) affine type: X is an infinite-dimensional affine space over a finite field F, and closure is F-affine closure.

Problem I. Are these the *only* examples of infinite, homogeneous, locally finite geometries?

The second problem concerns finite axiomatisability in logic, which also requires a little background for the nonlogician. Let L be a (countable) collection of symbols R_i, for $i \in I$, each of a specified finite "arity" $k(i)$, say. R_i will "stand for" an $k(i)$-ary relation. By a first-order L-sentence we mean an expression of the form $(Q_1 x_1) \cdots (Q_n x_n)(\Phi)$, where each Q_j is $\forall$ or $\exists$, and Φ is a finite Boolean combination of expressions, each of the form $R_i(x_{j(1)}, \dots, x_{j(k(i))})$ for some $i \in I$ and $j(1), \dots, j(k(i)) \le n$. Here the Boolean connectives are "and", "or", and "not". So quantification is allowed over elements but not over relations.

For example, if L consists of a single binary relation symbol $<$, then we can express the statement "$<$ is a linear ordering" by a finite collection of L-sentences. By an *L-structure*, M, say, we mean a set X (the underlying set of M) equipped with a $k(i)$-ary relation S_i on X for each $i \in I$. S_i is the *interpretation* of R_i. We usually assume that L contains a distinguished binary symbol R_0 which is always interpreted as *equality.* Often the underlying set X of M is also denoted by M. Any L-sentence σ will then be true or false in the structure M. Let Σ be a set of L-sentences (also often called an *L-theory*). We say that M is a *model of* Σ if every $\sigma \in \Sigma$ is true in M.

Let κ be an infinite cardinal. Σ is said to be *κ-categorical* if

(i) every model M of Σ is infinite (namely, has infinite underlying set), and

(ii) whenever $M_1 = (M_1, S_i)_{i\in I}$ and $M_2 = (M_2, T_i)_{i\in I}$ are models of Σ, each of cardinality κ, then M_1 is isomorphic to M_2; namely, there is a bijection between M_1 and M_2 which takes S_i to T_i for each $i \in I$. Σ is said to be *uncountably categorical* if Σ is κ-categorical for some uncountable κ and *totally categorical* if Σ is κ-categorical for all infinite κ. The second problem, possibly raised by Vaught, is

Problem II. Do there exist L and a *finite* set Σ of L-sentences such that Σ is totally categorical? Or, in usual parlance, does there exist a finitely axiomatisable totally categorical theory?

In the book under review, Zilber gives proofs of:

Theorem I. *Any infinite, homogeneous, locally finite geometry is degenerate, projective, or affine.*

Theorem II. *There is no finitely axiomatisable, totally categorical theory.*

Thus Problem I is answered positively and Problem II negatively. In fact, Zilber presents two proofs of Theorem II. The first uses Theorem I, and the second uses instead a difficult number-theoretic result due to Siegel. Both proofs depend on an analysis of the fine structure of models of uncountably categorical theories. The point of view, techniques, and results in Zilber's book have given rise to a new area in model theory, often called geometrical stability theory, and have also, in my opinion, substantially altered the way we view model theory and its relation to other areas of mathematics.

The modern systematic study of first-order logic is connected with attempts in the early part of this century to axiomatise mathematics in some "simple" fashion. The impossibility of a recursive such axiomatisation was shown by Godel. On the other hand, there exist reasonably interesting chunks of mathematics which can be recursively axiomatised. For example, we can consider the field of complex numbers as a structure (in the sense of model theory) consisting of the underlying set of complexes together with two ternary relations for the graphs of addition and multiplication. Let Σ_C be the set of first-order sentences true in this structure. Let ACF_0 be the axioms for field theory together with the (infinite, recursive) set of sentences expressing that the field is algebraically closed and of characteristic zero. Then ACF_0 is a subset of Σ_C, and elementary model theory shows that every sentence in Σ_C is a *logical consequence* of ACF_0. Similar results hold for the fields of real numbers and p-adic numbers. A natural further question to ask is whether some given structure M (a set equipped with certain relations) can be characterised *up to isomorphism* by the set Σ of first-order sentences true in it, namely, whether

Σ is *absolutely* categorical. The compactness and completeness theorems for first-order logic show this to be impossible unless M is finite. This is simply a cardinality question. If Σ is a set of sentences with an infinite model, then Σ will have models in all sufficiently large cardinalities. Thus one is led to the notion of λ-categoricity, mentioned earlier. This notion is clearly nonvacuous and moreover for interesting reasons. For example, any model of ACF_0 is (by definition) an algebraically closed field of characteristic 0 and is thus determined up to isomorphism by its transcendence degree over $\mathbf{Q}$. Clearly, then, AFC_0, and thus also Σ_C, is κ-categorical for *all* uncountable cardinals κ (but is *not* ω-categorical).

The starting point of modern stability theory was Morley's Theorem [M], proved in 1963: if the set of first-order sentences Σ is λ-categorical for some uncountable λ, then Σ is λ-categorical for all uncountable λ. Among the notions introduced by Morley was a rank (Morley rank) that could be attached to sets of first-order formulas in the context of an uncountably categorical theory Σ. Subsequently Shelah initiated a brilliant and successful line of research around trying to classify the possible functions $I(\Sigma, -)$, where $I(\Sigma, \kappa) =$ number of models of Σ of cardinality κ, up to isomorphism, and Σ is a first-order theory. The resulting subject was called classification theory, expounded in [Sh]. Among the notions developed by Shelah was that of a *stable theory*, a far-reaching generalisation of the notion of uncountably categorical theory. In the context of stable theories, an enormous technical machinery was built up, in particular, a notion of *independence* in models, again generalising that developed by Morley. This is what is called stability theory. In the meantime, Baldwin and Lachlan [BL] gave another proof of Morley's theorem, in which the "fine structure" of models of uncountably categorical theories was somewhat more in evidence. In particular, it was shown how a model M of an uncountably categorical theory is determined in a specific model-theoretic fashion from a "1-dimensional" subset of M. Zilber's line of research took off from the work of Baldwin and Lachlan (and also Palyutin [P]), with a quite different emphasis from Shelah's.

In order to describe in more detail these fine structure notions as well as Zilber's work, let us take a slightly different point of view of structures and theories which is nevertheless equivalent to the previous definitions.

By a structure M we now mean an infinite set (also called M) together with a certain (countable) collection $D_0(M)$ of distinguished subsets of various cartesian powers M^n of M, with the feature that $D_0(M)$ contains the set M; $D_0(M)$ contains the diagonal in M^2; and $D_0(M)$ is closed under Boolean combinations, Cartesian products, and projections $M^{n+1} \to M^n$. The sets in $D_0(M)$ will be called the $\varnothing$-*definable* sets in M.

Associated to M is then a language L, which has a symbol R_X for each $X \in D_0(M)$ (where R_X is k-ary if $X \subseteq M^k$). The *theory of* M, $\mathrm{Th}(M)$, is then the set of all first-order L-sentences true in M (under the natural interpretation). We call M uncountably categorical if $\mathrm{Th}(M)$ is uncountably categorical. Attached to M is also a larger class $D(M)$ of subsets of the various M^n, the *definable* sets. $D(M)$ is defined to be the collection of $X \subseteq M^n$ (n varying) such that for *some* $k \geq 0$, $Y \subseteq M^{n+k}$ with $Y \in D_0(M)$, and $\mathbf{a} \in M^k$, $X = \{\mathbf{b} \in M^n : (\mathbf{b}, \mathbf{a}) \in Y\}$. In fact, $X \in D(M)$ is said to be A-

definable (where A is some subset of M) if Y and $\mathbf{a}$ can be chosen as above with $\mathbf{a}$ from A. Under the assumption that M is uncountably categorical, it was shown that to each $X \in D(M)$, a dimension $\dim(X)$ can be assigned with certain characteristic properties. In fact, Morley showed just that the dimension function is ordinal valued, and Baldwin subsequently showed it had to be integer valued. In any case, the characteristic property is: $\dim(X) \geq 0$ for all $X \in D(M)$, and $\dim(X) \geq n+1$ if there is an infinite pairwise disjoint set $\{X_i : i < \omega\}$ of members of $D(M)$, such that for each i, $X_i \subseteq X$ and $\dim(X_i) \geq n$. This dimension function was traditionally called Morley rank. We thus have a certain category associated to M: the category of definable sets and definable maps between definable sets (where a map $f: X \to Y$ is said to be definable if its graph is). Such a category has a formal resemblance to natural categories in mathematics, such as the category of algebraic varieties and rational maps or the category of topological manifolds and continuous maps. For example, if $\dim(Y) = n$, $f: X \to Y$ is a definable surjective map, and for each $a \in Y$ $\dim(f^{-1}(\{a\}) = k)$, then $\dim(X) = n+k$. (The level of generality at which we are working, however, should be understood: M is an *arbitrary* structure (in the above sense) whose theory is uncountably categorical.)

In the hands of Zilber this resemblance is exploited brilliantly. In particular, Zilber exhibits the role played by group objects in the category. It was already known that M (our given uncountably categorical structure) contains a definable subset, say, D, with $\dim(D) = 1$ and that M is "determined" in some technical sense by D (specifically, M is "prime and minimal" over D). Zilber gives a precise mathematical form to this determination, showing in Chapter V that M is built from D by a finite sequence of "fibre bundles with transitive structure groups" in the sense of our category. For our purposes let us call (X, Y, f) a fibre bundle if X, Y are definable sets, f is a definable map from Y onto some definable set $Z \subseteq X^k$, and there is a definable family $\{G_a : a \in Z\}$ of definable groups, each a subset of X^n (for some n) and a definable family $\{g_a : a \in Z\}$ of definable *transitive* actions of G_a on $F_a = f^{-1}(a)$. Zilber proves (essentially) that there is a finite sequence of fibre bundles $(X_0, Y_0, f_0), \ldots, (X_n, Y_n, f_n)$ with $X_0 = D$ and $Y_n = M$. This is what he calls the Ladder Theorem. If all the structure groups (and thus also fibres) appearing in this sequence are finite, then elements of M correspond, up to finite, with k-tuples from D for some k (we say M is contained in the algebraic closure of D). But there *do* exist uncountably categorical structures where some of the structure groups have to be infinite, an example being the uncountably categorical (in fact, totally categorical) structure $((\mathbf{Z}/4\mathbf{Z})^{(\omega)}, +)$. As an aside, let us remark that the structure groups in the Ladder Theorem make their appearance as Galois groups, namely, groups of automorphisms of definable sets. The manner in which they originate and the proof of their existence are identical to the way in which differential Galois groups arise as algebraic groups in differential algebra, although the differential algebraic context is beyond that of uncountable categoricity.

We can now return to the original problems posed above and examine their connection with the above picture of an uncountably categorical structure. First, what is the connection with homogeneous geometries? We need a small definition. For A some subset of M and $a \in M$, define a to be in the *algebraic* *closure* of A, $a \in \mathrm{acl}(A)$ if there is some A-definable set X, such that X is finite and $a \in X$. Suppose D to be some $\varnothing$-definable 1-dimensional subset of M, which is in addition *irreducible*, in the sense that D cannot be partitioned into two infinite definable sets. (In fact, any 1-dimensional definable set in M can be written as a finite union of such irreducible 1-dimensional sets.) Let us define a closure operation cl(-) taking subsets of D to subsets of D, by $\mathrm{cl}(A) = \mathrm{acl}(A) \cap D$. It then turns out that (D, cl) satisfies properties (iii), (iv), and (v) in the definition of a geometry; we call (D, cl) a *pregeometry.* In fact, irreducibility of D implies that (D, cl) is a *homogeneous pregeometry.* We can attach a *homogeneous geometry* (D', cl') to D by simply quotienting out D-$\mathrm{cl}(\varnothing)$ by the equivalence relation $x \in \mathrm{cl}(y)$. This equivalence relation will not in general be definable. One of Zilber's fundamental insights was that these geometries should be studied—if possible, classified—and then the information used to understand the structure M.

Let us now assume that the uncountably categorical structure M is also *ω-categorical.* This imposes certain finiteness conditions on $D(M)$. For example, for each $n < \omega$ the set of $\varnothing$-definable subsets of M^n is *finite*, and, as a consequence, for every finite subset A of M, $\mathrm{acl}(A)$ is also finite. It follows that the pregeometry (D, cl) mentioned in the previous paragraph is *locally finite*, as is the associated geometry (D', cl'). It should be mentioned that under the ω-categoricity assumption the quotienting operation taking D to D' is *definable*, whereby D', and even (D', cl'), can be considered as a definable object in M too. Thus (D', cl') is a locally finite, homogeneous geometry, whereby Theorem I applies.

In fact, in proving Theorem I, there is no loss in generality in assuming that the geometry (D, cl) in the hypothesis of Theorem I *actually originates* from a 1-dimensional set in a totally categorical structure M in the way just described. The theorem is proved by a beautiful combination of combinatorial and model-theoretic methods. It is shown that if the theorem fails, then a certain incidence structure, called a *pseudoplane*, is definable in M. A pseudoplane is a kind of fuzzy projective plane: it consists of two infinite sets, P (points) and L (lines) together with an incidence relation $I \subseteq P \times L$, such that any point lies on infinitely many lines (and dually) and two distinct points lie on only finitely many common lines (and dually). On the other hand, the "local finiteness" resulting from ω-categoricity enables Zilber to assign certain polynomials with rational coefficients to definable sets in M. Essentially, if X is a definable set, then $p_X(x)$ is the polynomial such that whenever A is an algebraically closed subset of D of cardinality m, then $p_X(m)$ equals the cardinality of $X \cap \mathrm{acl}(A)$. Some highly nontrivial counting arguments involving the pseudoplane allow further control over the structure to be obtained, resulting in the existence of a definable 1-dimensional group G. The definable sets in this group are studied in detail, using the information gained earlier. It is shown that there is a definable vector space structure on G (over a finite field F) such that every definable subset of G^k is a Boolean combination of sets defined by linear equations. The geometry attached to the 1-dimensional set G must then be projective over F, which suffices to show that the geometry on D is projective or affine over F, proving Theorem I.

Theorem I thus yields *local* information about a totally categorical structure

M, in that it describes quite accurately the structure of 1-dimensional definable sets in M. Zilber is able to deduce global information from this, in particular, that any group definable in M is abelian-by-finite. This is then used to show that in the Ladder Theorem the structure groups (and thus the fibres on which they act) can all be chosen to be of dimension *at most one*. With this information Zilber proceeds to prove Theorem II. Nonfinite axiomatisability of $\Sigma = \mathrm{Th}(M)$ is proven by showing that for every sentence σ of Σ, there is a *finite* subset A of M in which σ is *true* (when quantifiers are restricted to A). A will be a sufficiently large finite "approximation" to M with enough symmetries. If $M = D$, the basic features of the geometry on D are enough to find A. Otherwise, induction on the length of the sequence of fibre bundles needed to produce M from D, together with the above-mentioned bound on the dimensions of fibres, plus further use of Zilber's polynomials allow A to be constructed.

In the last two paragraphs we have been discussing *totally categorical* structures. However, the "local-to-global" conclusions are also valid for *uncountably* categorical M. Given a one-dimensional definable set D in M, we have seen that the algebraic closure operation yields a pregeometry on D which we called (D, cl). This pregeometry gives rise to a lattice (the lattice of closed sets), and a crucial dichotomy is whether this lattice is *modular* or not. In fact, the more important question is whether or not this lattice is *locally* modular: local modularity means modularity after quotienting by some nontrivial closed set. Local modularity of some (all) 1-dimensional sets in M is equivalent to the nonexistence of definable pseudoplanes. Local modularity of (D, cl) again yields (as Zilber shows in Chapter II) global consequences for M, such as all definable groups being abelian-by-finite. The content of Theorem I is that all 1-dimensional sets in totally categorical structures are locally modular. However, there do exist nonlocally modular 1-dimensional sets outside the ω-categorical situation. An example is the field of complex numbers. It is worthwhile elucidating the geometric content of the locally modular/nonlocally modular dichotomy. Given a 1-dimensional set D, we can view 1-dimensional definable subsets of $D \times D$ as "curves" over D. Nonmodularity of the geometry attached to D is equivalent to the existence of an n-dimensional definable family of such curves for some $n \geq 2$, namely, the existence of "nonlinear" curves.

Finally, connections with other work should be mentioned. Subsequent to Zilber's announcement of his proof, Theorem I was proved by various other people using differing methods. Cherlin in [CHL] proved it using the classification of finite simple groups, specifically, the resulting classification of 2-transitive finite permutation groups. Evans [E] proved it using purely combinatorial methods. The most elegant proof is, for me, due to Hrushovski [H3], where model theory almost completely replaces the combinatorics.

In [CHL] Theorem II was generalised to a wider class of structures, those that are both ω-categorical and ω-stable. In our outline above the reader may have noticed that Theorem II was proved by showing that any sentence in Σ is true in some *finite* structure. This suggested the conjecture that totally categorical theories are finitely axiomatisable modulo the infinite collection of sentences "there are infinitely many elements". A special case of this conjecture was proved by Ahlbrandt and Ziegler [AZ] using a combinatorial fact about projective spaces over finite fields. The proof was generalised by Hrushovski [H2] to give the full conjecture.

Work of Hrushovski [H1] has also yielded more information on 1-dimensional sets which are not necessarily ω-categorical. In particular, if the attached geometry is modular and nondegenerate, then a 1-dimensional group can be definable; and the definable structure on this group is essentially just vector space structure with respect to some division ring.

Several model-theorists, most notably Ehud Hrushovski, have developed the insights of Zilber far beyond the uncountably categorical context and even beyond the context of stable theories. Among such developments the existence of definable groups and general analogues of the modular/nonmodular dichotomy have been crucial and have had substantial impact back on classification theory á la Shelah.

Zilber has succeeded in his book in showing how linear geometry can be recovered from model theory (this is the content of Theorem I). Another question, which has been an abiding concern of Zilber, is whether *algebraic* geometry can also be so recovered. The technical form this question takes is: given a 1-dimensional set D (in, say, an uncountably categorical structure), if the geometry attached to D is nonlocally modular, must there be an algebraically closed field definable? Although this turned out to be false in full generality, it has recently been shown to be true under the assumption that the definable subsets of D^n (various n) have a certain topological character, analogous to the Zariski topology [HZ].

Although there are other concise treatments of the subject matter of this book in print, or soon to be in print, it is well worth the effort to read Zilber's monograph for its classical point of view, freshness of style, and richness of ideas.

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