ON THE BUSEMANN-PETTY PROBLEM
CONCERNING CENTRAL SECTIONS
OF CENTRALLY SYMMETRIC CONVEX BODIES

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Abstract. We present a method which shows that in $\mathbb{E}^3$ the Busemann-Petty problem, concerning central sections of centrally symmetric convex bodies, has a positive answer. Together with other results, this settles the problem in each dimension.

In [BP], Busemann and Petty asked the following question, which resulted from reformulating a problem in Minkowskian geometry. Suppose $K_1$ and $K_2$ are convex bodies in $n$-dimensional Euclidean space $\mathbb{E}^n$ which are centered (centrally symmetric with center at the origin) and such that

$$\lambda_{n-1}(K_1 \cap u^\perp) \leq \lambda_{n-1}(K_2 \cap u^\perp),$$

for all $u$ in the unit sphere $S^{n-1}$. Then is it true that

$$\lambda_n(K_1) \leq \lambda_n(K_2)?$$

(Here $u^\perp$ denotes the hyperplane through the origin orthogonal to $u$, and $\lambda_k$ denotes $k$-dimensional Lebesgue measure, which we identify throughout with $k$-dimensional Hausdorff measure.)

The question, now generally known as the Busemann-Petty problem, has often appeared in the literature. More than thirty years ago, Busemann gave the problem wide exposure in [B2] and Klee posed it again in [K]. The problem attracted the attention of those working in the local theory of Banach spaces; see, for example, the paper [MP, p. 99] of Milman and Pajor. It surfaces again in Berger’s article [Be, p. 663], and it is also stated in the books of Burago and Zalgaller [BZ, p. 154]; Croft, Falconer, and Guy [CFG, Problem A9, p. 22]; and Schneider [S, p. 423].

The problem has an interesting history. Using a clever probabilistic argument, Larman and Rogers [LR] proved that the answer, surprisingly, is negative in $\mathbb{E}^n$ for $n \geq 12$. Later, Ball [B] applied his work on maximal sections of a cube to obtain a negative answer for $n \geq 10$, where $K_1$ is a centered cube and $K_2$ a centered ball of suitable radius. Giannopoulos [Gi] improved this negative result to $n \geq 7$ by using an appropriate cylinder for $K_1$ instead of a cube. Independently, Bourgain [Bo] showed that the same result can be achieved by taking $K_1$ to be a suitable arbitrarily small perturbation of a centered ball;
Bourgain also proved that his method will not work in $\mathbb{R}^3$. A further improvement was made by Papadimitrakis [P] and the author [Gl], independently, by demonstrating that the answer is negative for $n \geq 5$, when $K_1$ is a centered cylinder. The most recent negative answer was obtained, for $n \geq 4$ and $K_1$ a centered cube, by Zhang [Z1], [Z2]. For $4 \leq n \leq 6$, the existence of a suitable $K_2$ can be proved, though it seems likely that for these values of $n$, $K_2$ cannot be a ball, regardless of the choice of $K_1$. Other papers on the problem related to those mentioned above include [GR] and [T].

We shall outline a solution for $n = 3$, thereby settling the problem in each dimension. Against the background of the results above, the positive answer for $n = 3$ is unexpected. It is also especially interesting from the point of view of geometric tomography, in which one attempts to obtain information about a geometric object from data concerning its sections or projections. Geometric tomography has connections with functional analysis and possible applications to robot vision and stereology (see, for example, [BL, ES, GV, MP, W]).

A few positive results are already known. The case $n = 2$ is trivial. Busemann and Petty themselves noted in [BP] that the Busemann intersection inequality (see [Bl, (4), p. 2]) may be applied to obtain a positive answer when $K_1$ is a centered ellipsoid. Lutwak [L] obtained an important generalization of this fact by showing that the same is true whenever $K_1$ is a member of a certain class of bodies which he called intersection bodies.

We shall explain Lutwak’s result in some detail, since it is an essential ingredient in our method. A set $L$ in $\mathbb{R}^n$ is star shaped at the origin if it contains the origin and if every line through the origin meets $L$ in a (possibly degenerate) line segment. By a star body we mean a compact set $L$ which is star shaped at the origin and whose radial function

$$\rho_L(u) = \max\{c \geq 0 : cu \in L\},$$

for $u \in S^{n-1}$, is continuous on $S^{n-1}$. The star body $L$ is called the intersection body of another star body $M$ if

$$\rho_L(u) = \lambda_{n-1}(M \cap u^\perp),$$

for all $u \in S^{n-1}$. We write $L = IM$; it is clear that $L$ must be centered, and it is known (see [L]) that there is a unique centered star body $M'$ for which $L = IM'$.

A useful alternative viewpoint is provided by the spherical Radon transform. Suppose $g$ is a Borel function on $S^{n-1}$ and $f$ is defined by

$$f(u) = \int_{S^{n-1} \cap u^\perp} g(v) \, d\lambda_{n-2}(v),$$

for all $u \in S^{n-1}$; that is, $f(u)$ is the integral of $g$ over the great sphere in $S^{n-1}$ orthogonal to $u$. Then we write

$$f = Rg$$

and say that $f$ is the spherical Radon transform of $g$. Using the polar coordinate formula for volume, we see that a star body $L$ is the intersection body of some star body $M$ if and only if $\rho_L = Rg$ for some nonnegative continuous function $g$; just take $g = \rho_M^{n-1}/(n-1)$. 
Suppose $L_1$ is the intersection body of some star body and $L_2$ is an arbitrary star body, such that

$$\lambda_{n-1}(L_1 \cap u^⊥) \leq \lambda_{n-1}(L_2 \cap u^⊥),$$

for all $u \in S^{n-1}$. Then Lutwak's theorem (see [L, Theorem 10.1]) says that $\lambda_n(L_1) \leq \lambda_n(L_2)$.

It is worth noting that Lutwak has offered as an alternative and different definition of intersection body, a star body $L$ such that $\rho_L = R\mu$, where $\mu$ is an even finite Borel measure in $S^{n-1}$. A consequence of a result in [GLW] is that Lutwak's theorem still holds when $L_1$ is an intersection body in this wider sense of the term.

The class of intersection bodies is in a sense dual to the better-known class of projection bodies. The latter, which are just the centered zonoids, have been intensively studied and have many applications; see, for example, the articles of Bourgain and Lindenstrauss [BL], Goodey and Weil [GW], and Schneider and Weil [SW], or Schneider's book [S, Section 3.5]. In fact, the Busemann-Petty problem has a dual form in which sections are replaced by projections. This dual problem was solved, by Petty and Schneider independently, shortly after it was posed; the answer is negative for all $n > 2$. Lutwak's theorem concerning sections of star bodies is also dual to a corresponding one for projections of convex bodies, obtained by Petty and Schneider using tools from the Brunn-Minkowski theory (see [S, p. 422]). For sections, the extension from convex bodies to star bodies is not only natural but crucial. For example, it can be seen by direct calculation that a centered cylinder in $\mathbb{E}^3$ is the intersection body of a nonconvex centered star body; see [Gl, Remark 5.2(ii)].

The other known positive results are as follows. Hadwiger [H] and Giertz [Gie] independently showed that the question has an affirmative answer when $K_1$ and $K_2$ are coaxial centered convex bodies of revolution in $\mathbb{E}^3$. In [Gl, Theorem 5.1], it is proved that a centered convex body of revolution $K$ whose radial function $\rho_K$ belongs to $C^{\infty}(S^{n-1})$, the class of infinitely differentiable even functions on the unit sphere, is the intersection body of some star body when $n = 3$ or 4. (This result is re-proved in [Z2] by a different method.) Using Lutwak's theorem, it is easy to see that this implies that the Busemann-Petty problem has a positive answer in $\mathbb{E}^3$ or $\mathbb{E}^4$ whenever $K_1$ is a centered convex body of revolution. It has also been shown by Meyer [M] that the answer is positive in $\mathbb{E}^n$ provided that $K_1$ is a centered cross-polytope (the $n$-dimensional version of the octahedron).

We now sketch a proof that the Busemann-Petty problem has a positive answer in $\mathbb{E}^3$. The details will appear in [G2].

**Theorem.** The Busemann-Petty problem has an affirmative answer in $\mathbb{E}^3$.

**Sketch of the proof.** By approximating, we can assume that $\rho_{K_1} \in C^{\infty}_c(S^2)$ and that $K_1$ has everywhere positive Gaussian curvature. By Lutwak's theorem, it suffices to show that an arbitrary centered convex body $K$ in $\mathbb{E}^3$ with these additional properties is the intersection body of somestar body. (We make no attempt in this note to find the least restrictive additional conditions to impose on $K$.) This will be proved if there is a nonnegative function $g \in C(S^2)$ such that $\rho_K = Rg$. It is known that since $\rho_K \in C^{\infty}_e(S^2)$, a $g \in C^{\infty}_e(S^2)$ exists and
is unique. Let $u_0 \in S^2$. An inversion formula of Funk [F] gives

$$g(u_0) = \lim_{t \to -1} - \frac{1}{2\pi} \frac{d}{dt} \int_0^t \frac{x A_K(\sin^{-1}x)}{\sqrt{t^2 - x^2}} \, dx,$$

where $A_K(\phi)$ denotes the average of $\rho_K$ on the circle of latitude with angle $\phi$ from the north pole $u_0$. After some manipulation, one can obtain

$$2\pi g(u_0) = \rho_K(u_0) + \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\pi} \frac{\partial \rho_K(\theta, \phi)}{\partial \phi} \sec \phi \, d\phi \, d\theta,$$

where $(\theta, \phi)$ denotes the usual angles of spherical polar coordinates.

From $K$, construct a body $K_*$, called a Schwarz symmetral of $K$, as follows. Each horizontal section of $K_*$ is a disk whose center lies on the $z$-axis and whose area equals that of the horizontal section of $K$ of the same height. From the Brunn-Minkowski theorem, it follows that $K_*$ is a convex body of revolution, and our assumptions about $K$ allow us to conclude that $\rho_K \in C^1(S^2)$. One can then show that $\rho_K = R \bar{g}$, for some $\bar{g} \in C(S^2)$, and that equation (1) holds when $g$ and $\rho_K$ are replaced by $\bar{g}$ and $\rho_K$, respectively. Moreover, the argument of [Gl, Theorem 5.1] proves that $\bar{g}$ is nonnegative.

The final step of the proof involves applying a cylindrical transformation to equation (1). Once this is done, it can be seen that $g(u_0) = \bar{g}(u_0)$, and therefore $g(u_0) \geq 0$, as required. □

References


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