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Natural operations in differential geometry, by Ivan Kolář, Peter W. Michor, and Jan Slovák. Springer-Verlag, Berlin, Heidelberg, and New York, 1993, vi + 434 pp., \$89.00. ISBN 0-387-56235

This book represents an unusual approach to differential geometry (DG). Let me begin with a few brief quotes from the preface about its aims.

> *First,* it should be a monographical work on natural bundles and natural operators in DG. This is a field which every DG-er has met several times, but which is not treated in detail in one place.
>
> *Second,* this book tries to be a rather comprehensive textbook on all basic structures from the theory of jets which appear in different branches of DG. ... Even though Ehresmann in his original papers ... underlined the conceptual meaning, ... jets have been mostly used as a purely technical tool.
>
> *Third,* in the beginning of this book we give an introduction to the fundamentals of DG ... which stresses naturality and functoriality from the beginning. ... A specific feature of the book is that the authors are interested in general points of view to different structures in DG.

So, there you have it. This is not a textbook on Riemannian geometry or connections. It is not a "first course"; rather, it tries to fill the same need that was filled, sixty years ago, by Veblen and Whitehead's *Foundations of differential geometry.* It is in DG what Halmos's *Naive set theory* is in its field. It fills the gaps that are usually left, and answers questions that remain unanswered, in the more traditional treatments that show students a narrow path to some selected highlights.

At the time of Veblen and Whitehead, but also Eisenhart, Schouten, and other contemporaries, a tangent vector to a manifold was defined by the manner of transformation of its components. The same held for covectors, tensors, parameters of a connection, etc. In fact, the transformation law was what defined the type of *geometric object* in question. The behavior of these objects under point transformations (local diffeomorphisms) was also described in terms of local coordinates.

During the "anti-index revolution" of the 1950s, tangent vectors were seen as derivations on the ring of functions, and the other tensors were defined in terms of these, as duals, tensor products, and the like. There seldom was any need to prove invariance under coordinate transformations, since the definitions were intrinsic = coordinate-free.

There was far more to the "anti-index revolution" than just a matter of notation. The index-methods, which are basically clumsy, had been refined so far, with arrays of notational tricks, that DG had become virtually inaccessible to other mathematicians. Meanwhile, topologists, analysts, algebraists, etc., had realized the enormous unifying power of mappings as central building blocks of their fields. DG-ers were slow to follow, but the index-free methods did reestablish the position of DG in the mainstream of mathematics.

The 1950s were not the only time when attempts were made at index-free notation. Previous efforts earlier this century, however, were more or less artificial schemes to *suppress* indices. To understand the operations, you pretty much had to *think* the indices and then cleverly avoid writing them down. The "orgies of formalism" (a term due to Felix Klein) resulting from these methods were even worse than those of a consistent index method.

Today, vectors, tensors, etc., are seen as elements of vector bundles, whose total spaces are also manifolds. Thus, smoothness of sections in these bundles is just a special case of smoothness of mappings. The components and their transformations find their places as transition functions between the local trivializations of these bundles.

Nonvector quantities, such as connections, frames, jets, etc., are seen as more general fiber bundles. However, that does not provide a concept that distinguishes these bundles of "geometric objects" from the more general (vector) bundles whose construction is not as intimately related to the base manifold. The geometric objects are special, because only they automatically follow a point transformation.

This is where *natural bundles* come in. They are defined through *functors* that, for each type of geometric object, associate a fiber bundle with each manifold. To formalize this, we need two categories.

First, consider the category $\mathcal{M}f_m$ of m-dimensional (smooth) manifolds. Its morphisms are diffeomorphisms (into). Every open set of an m-manifold belongs to $\mathcal{M}f_m$: the theory is fundamentally a local one.

Second, consider the category $\mathcal{FM}$ of fibered manifolds (N, p, M), where M, N are manifolds and $p : N \to M$ is a surjective submersion. The inverse images $p^{-1}(x)$, for $x \in M$, are the fibers, N is the total space, and M the base space. The morphisms of $\mathcal{FM}$ are the fiber-preserving (smooth) maps. The base functor $B : \mathcal{FM} \to \mathcal{M}f$ assigns to each fibered manifold (N, p, M) its base manifold M and to each morphism in $\mathcal{FM}$ the induced map on the base spaces.

With these definitions, a *bundle functor* on $\mathcal{M}f_m$, or a *natural bundle* over m-manifolds, is a covariant functor $F : \mathcal{M}f_m \to \mathcal{FM}$ with these simple properties:

(1) (Prolongation) The base space of the fibered manifold FM is M itself.

(2) (Locality) If U is an open subset of M, then the total space of FU is $p^{-1}(U)$, the part of N above U.

Hidden in this definition (because of the use of the term *functor*) is the essence of natural bundles, namely, that every local diffeomorphism of the base spaces (morphism of $\mathcal{M}f_m$) "lifts" uniquely to the total spaces defined over them by F.

The definition of natural bundle comprises, in one blow, all the previously mentioned bundles of geometric objects. Of course, a flock of properties of natural bundles are to be proved. For example, a natural bundle is a fiber bundle (in the sense of Ehresmann), whose structure group is a subgroup of the diffeomorphism group of the fiber. The structure group is a homomorphic image of the group of germs of diffeomorphisms that fix the origin in $\mathbb{R}^m$. The sharp distinction between point transformations and coordinate transformations has disappeared: coordinate systems are simply local diffeomorphisms into $\mathbb{R}^m$, which belongs to $\mathcal{M}f_m$, and the functor F does the rest.

The concept of natural bundle was first formalized by the reviewer. It was little more than a definition, however, until some real theorems were proved. Work by Epstein and Thurston shows that natural bundles are of finite order (i.e., the structure group is a homomorphic image of finite-order jets of diffeomorphisms of $\mathbb{R}^m$ fixing the origin). In addition, their work shows that natural bundles have a natural smooth structure that automatically satisfies a regularity condition (not stated here) in the original definition of natural bundle. Basic to all of this is Peetre's Theorem, with a number of refinements. Other fundamental work, initiated by Palais and Terng, deals with the classification of natural vector bundles.

Natural bundles are a setting for a completely functorial approach to all tensor fields, connections, jets of sections, etc. The book under review does just that, thereby fulfilling one of its stated aims. If we compare this with Veblen and Whitehead's booklet, we see how much DG has progressed since then. So have the demands on the reader, who is expected to have mastered not only functions of several variables (implicit functions) but also large portions of multilinear algebra, some representation theory, as well as the basic concepts of category theory.

The pursuit of DG consists to a large extent of performing operations on sections of natural bundles. Connections are constructed from Riemann metrics, covariant derivatives are taken, Lie brackets of vector fields are formed, etc. These operations are natural; they commute with point transformations, yield smooth sections in natural bundles from the same, and have nonincreasing supports. All such natural operations are, as implied by Peetre-like theorems, of finite order and so induce natural transformations between corresponding jet bundles.

Peetre's Theorem depends heavily on smoothness. For example, the map $C^\infty(\mathbb{R}, \mathbb{R}) \to C^0(\mathbb{R}, \mathbb{R})$ given by $f \mapsto \sum_{k\geq 0} 2^{-k} \arctan \circ (d^k f/dx^k)$ is not reducible to finite order.

The general question of the classification of natural maps between natural bundles remains largely untouched. There are some useful general theorems and a number of interesting specific results of existence, nonexistence, and uniqueness. For example, there is only one natural operation that produces a connection from the 1-jets of a Riemann metric in spaces of dimension > 3. Similarly, the Chern forms generate all differential forms that can be naturally constructed from a symmetric connection. And, the only conformal natural forms on a Riemannian manifold are the Pontryagin forms (Gilkey's Theorem).

In connection-free contexts, the Frölicher-Nijenhuis (F-N) bracket is the "only" bilinear natural operator from a tangent vector-valued p- and q-form to the same kind of form of degree $p+q$. A similar result holds for the Schouten bracket of purely contravariant tensors. These include cases of Lie derivatives.

There are also statements regarding intrinsic structures on natural bundles. The tangent bundle (unlike the cotangent bundle) does not have a natural symplectic structure. The functor J^r, which assigns to manifolds M, N the bundle $J^r(M, N)$ of r-jets of maps $M \to N$, with base space $M \times N$, has no natural transformations $J^r \to J^r$ other than the contraction and the identity when $r > 1$; for $r = 1$, however, $J^1(M, N) \cong \mathrm{Hom}(TM, TN)$ and admits also the homothetic transformations. These are just a few of the simple-to-state conclusions.

In the spirit of generality, connections are initially introduced for fiber bundles without structure groups. Let Φ be a tangent vector-valued 1-form on the total space that projects each tangent space to its vertical subspace; then $\ker(\Phi)$ is the horizontal space of a connection. The connection is complete if every horizontal lifting of curves in the base space is global. The curvature is given by the F-N bracket $[\Phi, \Phi]$, whose values are vertical, while the Bianchi identity is simply $[\Phi, [\Phi, \Phi]] = 0$. A theorem generalizing those of Ambrose-Singer and of Nijenhuis states relationships between curvature and the Lie algebra of the holonomy group.

In a similar vein, Lie derivatives, though initially defined in the usual way, are later vastly generalized: $\tilde{\mathscr{L}}_{(\xi,\eta)}f$, for a map $f : M \to N$ and vector fields ξ on M and η on N, is defined as $\eta \circ f - Tf \circ \xi$. Subsequent specializations return to natural bundles, flows induced by vector fields, etc.

An entire chapter is devoted to *product preserving functors*, which are closely related to A. Weil's "infinitely near points". Another chapter relates these to more general bundle functors, including those with the so-called "point-property". The "flow-natural transformation" constitutes another major topic.

A final chapter is devoted to gauge natural bundles.

So much for the ideas that have been developed in this 400-page-plus volume on the foundations of contemporary differential geometry. It is extremely thorough and works out concepts in great detail. It is a great storehouse of information. In a way, "every" DG-er should be familiar with it. It provides a useful source for references to basic concepts. In any case, those who want to work on this type of material should consult it for problems solved or methods that may be applicable—or risk reinventing the wheel.

The authors have created an impressive monograph. I think it should have been named *Foundations of differential geometry*. Unfortunately, it was not designed to be a reference work or a textbook. It is hard to jump into the middle and find out what is going on. The organization is very tight, and many statements depend on definitions that – despite an elaborate system of section-numbering – are not easy to find. Also, there are no exercises. Exercises can provide a test of understanding and enable a reader, pencil and paper in hand, to participate in the development of ideas. In addition, exercises can replace some hints into directions that are not further pursued. Finally, if the solutions appear in the back, they can be used to prove technical lemmas and to relieve some of the monotony, with no loss of completeness.

In some subtle way, the readability is not as good as one would hope. The problem is not the English—it is perfect. Perhaps it is the frequent use of formulas where words (from a well-designed vocabulary) would do better. Also, the text is densely printed, so reading is rather tiring. By do-it-yourself TEX-ing, the authors have bypassed not only the expense of copy editors but also the experience these people can provide.

The material in this volume is mostly quite new. This becomes evident from reading the credits at the end of each chapter. It is also confirmed by the bibliography: more that 25 percent of the 300 plus entries date from 1980 or later. The authors' remark that the book "has been based on the common cultural heritage of Middle Europe" invites some further counting. Indeed, about 75 percent of the recent entries are by authors with some Central European connection. In addition to the authors, we see frequent mention of Janyška, Krupka, Mauhart, Mikulski, Zajtz, and others.

Despite the burst of Central European activity, I hope that the whole mathematical community will take note of this book. It has no geographic limitations or preferences. In fact, a number of the seminal ideas, predating 1980, came from other parts of the world. Without these, the book might not have been written.

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The Schwarz function and its generalization to higher dimensions, by Harold S. Shapiro. University of Arkansas Lecture Notes in the Mathematical Sciences, vol. 9, Wiley-Interscience, New York, 1992, xi+108 pp., $59.95. ISBN 0-471-57127-X

Looking at the contents of the book, one is tempted to suggest the alternative title: *Lectures on the Schwarz function and its potential in applications.* What is generalized to higher dimensions is not the Schwarz function itself but a related "Schwarz potential". Both the function and the potential play a role in a variety of topics. Prominent among the latter are so-called quadrature domains. These have been a special interest of the author and his coauthor, Dov Aharonov, and of, among others, Sakai [5, 6]. Perhaps of wider interest, at the hands of the author, the Schwarz function turns out to be a useful device in the study of various operators in complex analysis.

But what are the Schwarz function and the Schwarz potential?

Most mathematicians are familiar with H. A. Schwarz's reflection principle, which provides analytic continuation through reflection in a segment of the real axis. Specifically, let Ω be a domain in the upper half-plane $\operatorname{Im} z > 0$, part of whose boundary is a real open interval I. Let $f(z)$ be any holomorphic function on Ω which extends continuously to $\Omega \cup I$ and whose extension $f(x)$ to I is real. Then f can be continued analytically to the domain $\Omega \cup I \cup \overline{\Omega}$; the holomorphic continuation f^* is given by the functional equation

$$f^*(z) = \overline{f(\overline{z})}.$$

Here bars denote reflection in the real axis or complex conjugation.

It is perhaps less well known that Schwarz proved a similar result for analytic continuation across an arbitrary analytic boundary arc Γ. Just as in the case of the reflection $z \to \overline{z}$, the reflection function $R(z) = R_\Gamma(z)$ for an analytic arc Γ is antiholomorphic. Long ago a few mathematicians, D.-A. Grave (1895) in France and G. Herglotz (1914) in Germany, started to use the complex conjugate of Schwarz's reflection function which is holomorphic. However, it took Davis