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## CLOSED IDEALS OF THE ALGEBRA OF ABSOLUTELY CONVERGENT TAYLOR SERIES

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**ABSTRACT.** Let  $\Gamma$  be the unit circle,  $A(\Gamma)$  the Wiener algebra of continuous functions whose series of Fourier coefficients are absolutely convergent, and  $A^+$  the subalgebra of  $A(\Gamma)$  of functions whose negative coefficients are zero. If  $I$  is a closed ideal of  $A^+$ , we denote by  $S_I$  the greatest common divisor of the inner factors of the nonzero elements of  $I$  and by  $I^A$  the closed ideal generated by  $I$  in  $A(\Gamma)$ . It was conjectured that the equality  $I^A = S_I H^\infty \cap I^A$  holds for every closed ideal  $I$ . We exhibit a large class  $\mathcal{F}$  of perfect subsets of  $\Gamma$ , including the triadic Cantor set, such that the above equality holds whenever  $h(I) \cap \Gamma \in \mathcal{F}$ . We also give counterexamples to the conjecture.

### 1. INTRODUCTION

Let  $D$  be the open unit disk, let  $\Gamma$  be the unit circle, and let  $H^\infty = H^\infty(D)$  be the algebra of bounded holomorphic functions on  $D$ . Let  $A^+$  be the algebra of absolutely convergent Taylor series, i.e., the algebra of analytic functions  $f: D \rightarrow \mathbb{C}$  such that  $\|f\|_1 = \sum_{n=0}^{\infty} \frac{|f^{(n)}(0)|}{n!} < +\infty$ . Clearly,  $A^+$  is a subalgebra of the disc algebra  $\mathcal{U}(D)$  consisting of functions continuous on  $\bar{D}$  and holomorphic on  $D$ . Also, if we denote by  $A(\Gamma)$  the algebra of absolutely convergent Fourier series, equipped with the norm  $\|f\|_1 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|$ , and we identify  $f \in A^+$  with  $f|_\Gamma$ , we can write  $A^+ = \{f \in A(\Gamma) : \hat{f}(n) = 0 (n < 0)\}$ .

Let  $I \neq \{0\}$  be a closed ideal of  $A^+$ , let  $S_I$  be the greatest common divisor of the inner factors of all nonzero elements of  $I$ ,  $I^A$  the closed ideal of  $A(\Gamma)$  generated by  $I$ , and  $h(I) = \{z \in \bar{D} : f(z) = 0 (f \in I)\}$ . If  $E \subseteq \Gamma$  is closed, set  $I^+(E) = \{f \in A^+ : f|_E \equiv 0\}$ . It was proved long ago by Carleson [3] that  $I^+(E) = \{0\}$  for certain closed sets  $E$  of measure zero. The structure of those closed ideals  $I$  of  $A^+$  such that  $h(I)$  is finite or countable was described in 1972 by Kahane [12] and Bennett-Gilbert [2]. In this case,  $I = I^+(h(I) \cap \Gamma) \cap S_I \cdot H^\infty$ . Bennett-Gilbert conjectured in [2] that, in general,

**Conjecture 1.**  $I = I^A \cap S_I \cdot H^\infty$ .

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(This conjecture was also quoted by Kahane [12].) Similar conjectures have been verified by Beurling-Rudin [19] for  $\mathcal{U}(D)$ , by Taylor-Williams [20] for  $A^\infty(D) = \{f \in \mathcal{U}(D) : f^{(n)} \in \mathcal{U}(D) \ (n \geq 1)\}$ , by Korenblum [16] for  $A^p(D) = \{f \in \mathcal{U}(D) : f^{(n)} \in \mathcal{U}(D) \ (n \leq p)\}$ , and by Matheson [17] for

$$\Lambda_\alpha(D) = \{f \in \mathcal{U}(D) : \lim_{|t_1 - t_2| \rightarrow 0} \frac{|f(e^{it_1}) - f(e^{it_2})|}{|t_1 - t_2|^\alpha} = 0 \text{ uniformly}\}.$$

The purpose of this note is to describe recent progress concerning closed ideals of  $A^+$ .

The main results can be described as follows:

**Theorem 1.1** [6]. *Let  $p$  be an integer  $\geq 3$ . If  $h(I) \cap \Gamma \subseteq E_{1/p}$ , the perfect symmetric set of constant ratio  $\frac{1}{p}$ , then  $I$  satisfies Conjecture 1.*

**Theorem 1.2** [7]. *There exists a Kronecker set  $E$  and a closed ideal  $I$  of  $A^+$  such that  $S_I = 1$ ,  $h(I) = E$ , which does not satisfy Conjecture 1.*

Results for  $L^1(\mathbb{R}^+)$  which concern ideals whose hull is countable and are analogous to the results of Kahane and Bennett-Gilbert have been obtained by Nyman [18] and Gurarii [9, 10]. By using transfer methods due to Hedemalm [11], El Fallah [4] has proved certain versions of Theorem 1.1 and Theorem 1.2 for  $L^1(\mathbb{R}^+)$ .

## 2. DIVISION IDEALS

If  $I \neq \{0\}$  is a closed ideal of  $A^+$  and if  $f \in A^+$ , we denote by  $I(f) = \{g \in A^+ : fg \in I\}$  the *division ideal* associated with  $f$ . We will say that a closed subset  $E \subseteq \Gamma$  is a Carleson set if

$$\int_{-\pi}^{\pi} \log \left( \frac{2}{\text{dist}(e^{it}, E)} \right) dt < \infty.$$

This condition is necessary and sufficient for the existence of a nonzero  $f \in \Lambda_\alpha$  vanishing on  $E$ . It is also necessary and sufficient for the existence of an outer  $f \in A^\infty(D)$  such that  $f$  vanishes exactly on  $E$  and  $f|_E^{(n)} = 0$  ( $n \geq 1$ ); see [3, 20].

Some improvements of the methods used to discuss closed ideals of  $A^\infty(D)$  and  $A^+(D)$  lead to the following result (one must circumvent the fact that the outer part of an element of  $A^+$  does not necessarily belong to  $A^+$ ):

**Theorem 2.1.** (i) *If  $f \in A^+ \cap S_I H^\infty$ , then  $h(I(f)) \subseteq h(I) \cap \Gamma$ . Also, the positive singular measure which defines the inner factor of  $I(f)$  is nonatomic and vanishes on all Carleson sets.*

(ii) *If, further,  $f$  vanishes on  $h(I) \cap \Gamma$ , then  $h(I(f))$  is a perfect subset of  $\Gamma$ .*

We note that Theorem 2.1(ii) contains the results of Kahane and Bennett-Gilbert, for, if  $h(I)$  is countable, it implies that  $h(I(f)) = \emptyset$ .

The results and methods of [1, 17, 20] lead, also, to the following information:

**Theorem 2.2.** *Let  $E$  be a Carleson set, and denote by  $J_0^+(E)$  the closure in  $A^+$  of the set of elements of  $A^\infty(D)$  vanishing on  $E$  with all their derivatives. Then:*

(i)  *$J_0^+(E) \subseteq I$  for every closed ideal  $I$  of  $A^+$  such that  $S_I = 1$  and  $h(I) \subseteq E$ .*

(ii) *If  $\alpha > \frac{1}{2}$  and  $f \in \Lambda_\alpha(D) \cap I^+(E)$ , then  $f \in J_0^+(E)$ .*

### 3. WHEN THE CONJECTURE WORKS

The following lemma, related to the Katznelson-Tzafriri theorem for contractions [5, 14], is the key to our positive results concerning the Bennett-Gilbert conjecture. The  $w^*$  topology discussed below is defined by considering  $A^+$  as the dual of  $c_0$ .

**Lemma 3.1.** *Let  $I$  be a closed ideal of  $A^+$ , and let  $f \in I^A \cap A^+$ . Then  $I(f)$  is  $w^*$ -closed.*

Using the results of §2, we obtain:

**Theorem 3.2** [6]. *Let  $E \subseteq \Gamma$  be a Carleson set. If  $J_0^+(E)$  is  $w^*$ -dense in  $A^+$ , then  $I = I^A \cap S_I H^\infty$  for every closed ideal of  $A^+$  such that the perfect part of  $h(I) \cap \Gamma$  is contained in  $E$ .*

Thus, Theorem 1.1 follows from the fact that  $J_0^+(E_{1/p})$  is indeed  $w^*$ -dense in  $A^+$ .

### 4. WHEN THE CONJECTURE FAILS

If  $K$  is a Helson set, then  $I^+(K)$  is  $w^*$ -dense in  $A^+$  (this observation is an extension of [8, Theorem 4.5.2]). Also, if  $K$  is a Kronecker set and a Carleson set, then  $J_0^+(K)$  has no inner factor; so the closed ideal generated by  $J_0^+(K)$  consists of all functions of  $A(\Gamma)$  vanishing on  $K$ , since Kronecker sets satisfy synthesis [21]. So, if the ideal  $J_0^+(K)$  satisfies the Bennett-Gilbert conjecture, we must have  $J_0^+(K) = I^+(K)$ . A construction which has some relation with Kaufman's construction of a Helson set of multiplicity [15] gives the following result:

**Theorem 4.1** [7]. *Let  $E$  be a set of multiplicity. Then there exists a nonzero distribution  $\mu$  whose support is a Kronecker subset of  $E$  such that  $\hat{\mu}(n) \rightarrow 0$  as  $n \rightarrow -\infty$ .*

Theorem 1.2 follows from Theorem 4.1 applied to a Carleson set of multiplicity (for example, the perfect set  $E_\xi$  when  $\frac{1}{\xi}$  is not a Pisot number). In this case, if  $K$  is the support of the distribution given by Theorem 4.1, we have  $J_0^+(K)$  properly contained in  $I^+(K)$ , and it is even possible to show that there are  $2^{\aleph_0}$  distinct closed ideals between  $J_0^+(K)$  and  $I^+(K)$ , ideals which would be equal if the Bennett-Gilbert conjecture were true.

### 5. APPLICATIONS

The positive results about the Bennett-Gilbert conjecture give "strong uniqueness properties" of some closed subsets of  $\Gamma$ . For example, it follows from Theorem 2.1 that any distribution  $S$  supported by  $E_{1/p}$  such that  $\widehat{S}(n) \rightarrow 0$  as  $n \rightarrow \infty$  must be the zero distribution. Some stronger results involving hyperdistributions can be found in [7]. We present here an application of Theorem 1.1 to operator theory (an extension of the Beurling-Pollard method [13, p. 61] is involved in the proof).

**Theorem 5.1.** *Let  $T$  be a contraction on a Banach space. If  $\text{Sp } T \subseteq E_{1/p}$  and if*

$$\limsup_{n \rightarrow \infty} \frac{\log^+ \|T^{-n}\|}{n^\alpha} < +\infty \quad \text{where } \alpha < \frac{\log p - \log 2}{2 \log p - \log 2},$$

*then  $\sup_{n \geq 1} \|T^{-n}\| < +\infty$ .*

If we add to the hypotheses of Theorem 5.1 the assumption that  $\text{Sp } T$  is a Dirichlet set, we can conclude that  $T$  is an isometry. Analogous results hold for all  $E_\xi$  with  $\xi \in (0, \frac{1}{2})$  if we consider only Hilbert spaces. On the other hand, if  $E$  is a set of multiplicity (for example,  $E = E_\xi$  when  $\frac{1}{\xi}$  is not a Pisot number), then there exist contractions  $T$  such that  $\text{Sp } T \subseteq E$  and  $\|T^{-n}\|$  goes to infinity arbitrarily slowly as  $n$  goes to infinity.

Zarrabi [22] proved that if  $E \subseteq \Gamma$  is countable, every contraction on a Banach space  $X$  such that  $\text{Sp } T \subseteq X$  and  $\log^+ \|T^{-n}\|/n^{1/2} \rightarrow 0$  as  $n \rightarrow \infty$  is an isometry; but this property holds exclusively for countable sets, even if we suppose that  $X$  is a Hilbert space.

It follows from unpublished computations by M. Zarrabi, M. Rajoelina, and the first author that the constant  $\frac{\log p - \log 2}{2 \log p - \log 2}$  in Theorem 5.1 is the best possible.

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