RESEARCH ANNOUNCEMENTS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, Number 1, July 1994

CLOSED IDEALS OF THE ALGEBRA OF ABSOLUTELY CONVERGENT TAYLOR SERIES

J. ESTERLE, E. STROUSE AND F. ZOUAKIA

ABSTRACT. Let Γ be the unit circle, $A(\Gamma)$ the Wiener algebra of continuous functions whose series of Fourier coefficients are absolutely convergent, and A^+ the subalgebra of $A(\Gamma)$ of functions whose negative coefficients are zero. If I is a closed ideal of A^+, we denote by S_I the greatest common divisor of the inner factors of the nonzero elements of I and by I^A the closed ideal generated by I in $A(\Gamma)$. It was conjectured that the equality $I^A = S_I H^\infty \cap I^A$ holds for every closed ideal I. We exhibit a large class $\mathscr{F}$ of perfect subsets of Γ, including the triadic Cantor set, such that the above equality holds whenever $h(I) \cap \Gamma \in \mathscr{F}$. We also give counterexamples to the conjecture.

1. INTRODUCTION

Let D be the open unit disk, let Γ be the unit circle, and let $H^\infty = H^\infty(D)$ be the algebra of bounded holomorphic functions on D. Let A^+ be the algebra of absolutely convergent Taylor series, i.e., the algebra of analytic functions $f\colon D \to \mathbb{C}$ such that $\|f\|_1 = \sum_{n=0}^{\infty} \frac{|f^{(n)}(0)|}{n!} < +\infty$. Clearly, A^+ is a subalgebra of the disc algebra $\mathscr{U}(D)$ consisting of functions continuous on $\overline{D}$ and holomorphic on D. Also, if we denote by $A(\Gamma)$ the algebra of absolutely convergent Fourier series, equipped with the norm $\|f\|_1 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|$, and we identify $f \in A^+$ with $f_{|\Gamma}$, we can write $A^+ = \{f \in A(\Gamma) : \hat{f}(n) = 0 (n < 0)\}$.

Let $I \neq \{0\}$ be a closed ideal of A^+, let S_I be the greatest common divisor of the inner factors of all nonzero elements of I, I^A the closed ideal of $A(\Gamma)$ generated by I, and $h(I) = \{z \in \overline{D}\colon f(z) = 0 (f \in I)\}$. If $E \subseteq \Gamma$ is closed, set $I^+(E) = \{f \in A^+ : f_{|E} \equiv 0\}$. It was proved long ago by Carleson [3] that $I^+(E) = \{0\}$ for certain closed sets E of measure zero. The structure of those closed ideals I of A^+ such that $h(I)$ is finite or countable was described in 1972 by Kahane [12] and Bennett-Gilbert [2]. In this case, $I = I^+(h(I) \cap \Gamma) \cap S_I \cdot H^\infty$. Bennett-Gilbert conjectured in [2] that, in general,

Conjecture 1. $I = I^A \cap S_I \cdot H^\infty$.

Received by the editors June 25, 1993.
1991 *Mathematics Subject Classification*. Primary 46H10, 43A20; Secondary 43A46, 46F20.

(This conjecture was also quoted by Kahane [12].) Similar conjectures have been verified by Beurling-Rudin [19] for $\mathscr{U}(D)$, by Taylor-Williams [20] for $A^\infty(D) = \{f \in \mathscr{U}(D) : f^{(n)} \in \mathscr{U}(D)\ (n \geq 1)\}$, by Korenblum [16] for $A^p(D) = \{f \in \mathscr{U}(D) : f^{(n)} \in \mathscr{U}(D)\ (n \leq p)\}$, and by Matheson [17] for

$$\Lambda_\alpha(D) = \{f \in \mathscr{U}(D) : \lim \frac{|f(e^{it_1}) - f(e^{it_2})|}{|t_1 - t_2|^\alpha} = 0 \text{ uniformly as } |t_1 - t_2| \to 0)\}.$$

The purpose of this note is to describe recent progress concerning closed ideals of A^+.

The main results can be described as follows:

Theorem 1.1 [6]. *Let p be an integer ≥ 3. If $h(I) \cap \Gamma \subseteq E_{1/p}$, the perfect symmetric set of constant ratio $\frac{1}{p}$, then I satisfies Conjecture* 1.

Theorem 1.2 [7]. *There exists a Kronecker set E and a closed ideal I of A^+ such that $S_I = 1$, $h(I) = E$, which does not satisfy Conjecture* 1.

Results for $L^1(\mathbb{R}^+)$ which concern ideals whose hull is countable and are analogous to the results of Kahane and Bennett-Gilbert have been obtained by Nyman [18] and Gurarii [9, 10]. By using transfer methods due to Hedenmalm [11], El Fallah [4] has proved certain versions of Theorem 1.1 and Theorem 1.2 for $L^1(\mathbb{R}^+)$.

2. Division ideals

If $I \neq \{0\}$ is a closed ideal of A^+ and if $f \in A^+$, we denote by $I(f) = \{g \in A^+ : fg \in I\}$ the *division ideal* associated with f. We will say that a closed subset $E \subseteq \Gamma$ is a Carleson set if

$$\int_{-\pi}^{\pi} \log\left(\frac{2}{\operatorname{dist}(e^{it}, E)}\right) dt < \infty.$$

This condition is necessary and sufficient for the existence of a nonzero $f \in \Lambda_\alpha$ vanishing on E. It is also necessary and sufficient for the existence of an outer $f \in A^\infty(D)$ such that f vanishes exactly on E and $f^{(n)}_{|E} = 0\ (n \geq 1)$; see [3, 20].

Some improvements of the methods used to discuss closed ideals of $A^\infty(D)$ and $A^+(D)$ lead to the following result (one must circumvent the fact that the outer part of an element of A^+ does not necessarily belong to A^+):

Theorem 2.1. (i) *If $f \in A^+ \cap S_I H^\infty$, then $h(I(f)) \subseteq h(I) \cap \Gamma$. Also, the positive singular measure which defines the inner factor of $I(f)$ is nonatomic and vanishes on all Carleson sets.*

(ii) *If, further, f vanishes on $h(I) \cap \Gamma$, then $h(I(f))$ is a perfect subset of Γ.*

We note that Theorem 2.1(ii) contains the results of Kahane and Bennett-Gilbert, for, if $h(I)$ is countable, it implies that $h(I(f)) = \varnothing$.

The results and methods of [1, 17, 20] lead, also, to the following information:

Theorem 2.2. *Let E be a Carleson set, and denote by $J_0^+(E)$ the closure in A^+ of the set of elements of $A^\infty(D)$ vanishing on E with all their derivatives. Then*:

(i) *$J_0^+(E) \subseteq I$ for every closed ideal I of A^+ such that $S_I = 1$ and $h(I) \subseteq E$.*

(ii) *If $\alpha > \frac{1}{2}$ and $f \in \Lambda_\alpha(D) \cap I^+(E)$, then $f \in J_0^+(E)$.*

3. When the conjecture works

The following lemma, related to the Katznelson-Tzafriri theorem for contractions [5, 14], is the key to our positive results concerning the Bennett-Gilbert conjecture. The $w^\star$ topology discussed below is defined by considering A^+ as the dual of c_0.

Lemma 3.1. *Let I be a closed ideal of A^+, and let $f \in I^A \cap A^+$. Then $I(f)$ is $w^\star$-closed.*

Using the results of §2, we obtain:

Theorem 3.2 [6]. *Let $E \subseteq \Gamma$ be a Carleson set. If $J_0^+(E)$ is $w^\star$-dense in A^+, then $I = I^A \cap S_I H^\infty$ for every closed ideal of A^+ such that the perfect part of $h(I) \cap \Gamma$ is contained in E.*

Thus, Theorem 1.1 follows from the fact that $J_0^+(E_{1/p})$ is indeed $w^\star$-dense in A^+.

4. When the conjecture fails

If K is a Helson set, then $I^+(K)$ is $w^\star$-dense in A^+ (this observation is an extension of [8, Theorem 4.5.2]). Also, if K is a Kronecker set and a Carleson set, then $J_0^+(K)$ has no inner factor; so the closed ideal generated by $J_0^+(K)$ consists of all functions of $A(\Gamma)$ vanishing on K, since Kronecker sets satisfy synthesis [21]. So, if the ideal $J_0^+(K)$ satisfies the Bennett-Gilbert conjecture, we must have $J_0^+(K) = I^+(K)$. A construction which has some relation with Kaufman's construction of a Helson set of multiplicity [15] gives the following result:

Theorem 4.1 [7]. *Let E be a set of multiplicity. Then there exists a nonzero distribution μ whose support is a Kronecker subset of E such that $\hat{\mu}(n) \to 0$ as $n \to -\infty$.*

Theorem 1.2 follows from Theorem 4.1 applied to a Carleson set of multiplicity (for example, the perfect set E_ξ when $\frac{1}{\xi}$ is not a Pisot number). In this case, if K is the support of the distribution given by Theorem 4.1, we have $J_0^+(K)$ properly contained in $I^+(K)$, and it is even possible to show that there are $2^{\aleph_0}$ distinct closed ideals between $J_0^+(K)$ and $I^+(K)$, ideals which would be equal if the Bennett-Gilbert conjecture were true.

5. Applications

The positive results about the Bennett-Gilbert conjecture give "strong uniqueness properties" of some closed subsets of Γ. For example, it follows from Theorem 2.1 that any distribution S supported by $E_{1/p}$ such that $\widehat{S}(n) \to 0$ as $n \to \infty$ must be the zero distribution. Some stronger results involving hyperdistributions can be found in [7]. We present here an application of Theorem 1.1 to operator theory (an extension of the Beurling-Pollard method [13, p. 61] is involved in the proof).

Theorem 5.1. *Let T be a contraction on a Banach space. If $\operatorname{Sp} T \subseteq E_{1/p}$ and if*

$$\limsup_{n\to\infty} \frac{\log^+ \|T^{-n}\|}{n^\alpha} < +\infty \quad \textit{where } \alpha < \frac{\log p - \log 2}{2\log p - \log 2},$$

then $\sup_{n\geq 1} \|T^{-n}\| < +\infty$.

If we add to the hypotheses of Theorem 5.1 the assumption that $\mathrm{Sp}\,T$ is a Dirichlet set, we can conclude that T is an isometry. Analogous results hold for all E_ξ with $\xi \in (0, \frac{1}{2})$ if we consider only Hilbert spaces. On the other hand, if E is a set of multiplicity (for example, $E = E_\xi$ when $\frac{1}{\xi}$ is not a Pisot number), then there exist contractions T such that $\mathrm{Sp}\,T \subseteq E$ and $\|T^{-n}\|$ goes to infinity arbitrarily slowly as n goes to infinity.

Zarrabi [22] proved that if $E \subseteq \Gamma$ is countable, every contraction on a Banach space X such that $\mathrm{Sp}\,T \subseteq X$ and $\log^+ \|T^{-n}\|/n^{1/2} \to 0$ as $n \to \infty$ is an isometry; but this property holds exclusively for countable sets, even if we suppose that X is a Hilbert space.

It follows from unpublished computations by M. Zarrabi, M. Rajoelina, and the first author that the constant $\frac{\log p - \log 2}{2\log p - \log 2}$ in Theorem 5.1 is the best possible.

References

1. A. Atzmon, *Operators which are annihilated by analytic functions and invariant subspaces*, Acta. Math. **144** (1980), 27–63.
2. C. Bennett and J. E. Gilbert, *Homogeneous algebras on the circle: I-ideals of analytic functions*, Ann. Inst. Fourier Grenoble **22** (1972), 1–19.
3. L. Carleson, *Sets of uniqueness of functions regular in the unit circle*, Acta Math. **87** (1952), 325–345.
4. O. El Fallah, *Idéaux fermés de* $L^1(\mathbb{R}^+)$, Math. Scand. (1) **72** (1993), 120–130.
5. J. Esterle, E. Strouse, and F. Zouakia, *Theorems of Katznelson-Tzafriri type for contractions*, J. Funct. Anal. **94** (1990), 273–287.
6. ———, *Closed ideals of* A^+ *and the Cantor set*, J. Reine Angew. Math. (to appear).
7. J. Esterle, *Distributions on Kronecker sets, strong forms of uniqueness, and closed ideals of* A^+, J. Reine Angew. Math. (to appear).
8. C. C. Graham and O. C. McGehee, *Essays in commutative harmonic analysis*, Springer-Verlag, Berlin, Heidelberg, and New York, 1979.
9. V. P. Gurarii, *Spectral synthesis of bounded functions on the half axis*, Funct. Anal. Prilozhen **4** (1969), 34–48.
10. ———, *Harmonic analysis in spaces with weight*, Trans. Moscow Math. Soc. **35** (1979), 21–75.
11. H. Hedenmalm, *A comparison between the closed ideals in* l^1_ω *and* L^1_ω, Math. Scand. **58** (1986), 275–300.
12. J. P. Kahane, *Idéaux fermés dans certaines algèbres de fonctions analytiques*, Actes Table Ronde Int. C. N. R. S. Montpellier, Lecture Notes in Math., vol. 336, Springer-Verlag, Berlin, Heidelberg, and New York, 1973, pp. 5–14.
13. ———, *Series de Fourier absolument convergentes*, Ergeb. Math. Grenzgeb. (3), vol. 50, Springer-Verlag, Berlin, Heidelberg, and New York, 1970.
14. Y. Katznelson and L. Tzafriri, *On power bounded operators*, J. Funct. Anal. **68** (1986), 313–328.
15. R. Kaufman, *M-sets and distributions*, Asterisque **5** (1973), 225–230.
16. B. I. Korenblum, *Closed ideals in the ring* A^n, Funct. Anal. Appl. **6** (1972), 203–214.
17. A. L. Matheson, *Closed ideals in rings of analytic functions satisfying a lipschitz condition*, Lecture Notes in Math., vol. 604, Springer-Verlag, Berlin, Heidelberg, and New York, 1976, pp. 67–72.
18. B. Nyman, *On the one dimensional translation group and semigroup in certain function spaces*, Thesis, Uppsala, 1950.

19. W. Rudin, *The closed ideals in an algebra of analytic functions*, Canad. J. Math. **9** (1957), 426–434.

20. B. A. Taylor and D. L. Williams, *Ideals in rings of analytic functions with smooth boundary values*, Canad. J. Math. **22** (1970), 1266–1283.

21. N. Varopoulos, *Sur les ensembles parfaits et les series trigonometriques*, C. R. Acad. Sci. Paris Sér. I. Math. **260** (1965), 3831–3834.

22. M. Zarrabi, *Contractions à spectre denombrable et propriétés d'unicité forte des fermés denombrables du cercle*, Ann. Inst. Fourier (1) **43** (1993), 251–263.

(J. Esterle and E. Strouse) UFR de Mathematiques et Informatique, Université de Bordeaux I, 351 cours de la Liberation, 33405 Talence, France

E-mail address, E. Strouse: `strouse@ecole.ceremab.u-bordeaux.fr`

(F. Zouakia) Department de Mathematiques, ENS Takkadoum, BP5118 Rabat, Morocco