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A THEORY OF CHARACTERISTIC CURRENTS ASSOCIATED WITH A SINGULAR CONNECTION

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ABSTRACT. This note announces a general construction of characteristic currents for singular connections on a vector bundle. It develops, in particular, a Chern-Weil-Simons theory for smooth bundle maps $\alpha : E \to F$ which, for smooth connections on E and F, establishes formulas of the type

$$\phi = \mathrm{Res}_\phi \Sigma_\alpha + dT.$$

Here ϕ is a standard charactersitic form, Res_ϕ is an associated smooth "residue" form computed canonically in terms of curvature, Σ_α is a rectifiable current depending only on the singular structure of α, and T is a canonical, functorial transgression form with coefficients in L^1_{loc}. The theory encompasses such classical topics as: Poincaré-Lelong Theory, Bott-Chern Theory, Chern-Weil Theory, and formulas of Hopf. Applications include: a new proof of the Riemann-Roch Theorem for vector bundles over algebraic curves, a C^∞-generalization of the Poincaré-Lelong Formula, universal formulas for the Thom class as an equivariant characteristic form (i.e., canonical formulas for a de Rham representative of the Thom class of a bundle with connection), and a Differentiable Riemann-Roch-Grothendieck Theorem at the level of forms and currents. A variety of formulas relating geometry and characteristic classes are deduced as direct consequences of the theory.

1. INTRODUCTION

Our purpose is to announce a generalized Chern-Weil theory for singular connections on a vector bundle, particularly those which arise from bundle maps. Specifically, if E and F are smooth vector bundles with connection over a manifold X, the theory associates to each homomorphism $\alpha : E \longrightarrow F$ a family of d-closed characteristic currents on X defined canonically in terms of the curvature of the bundles and the singularities of the map α.

One can focus attention either on E or on F (and retrieve, when $\alpha \equiv 0$, the standard theory for E or F). Suppose the focus is on F, and fix a *characteristic polynomial* ϕ, i.e., an Ad-invariant polynomial on the Lie algebra of the structure group of F. The new theory associates to ϕ a d-closed current $\phi(\Omega_{F,\alpha})$ which is defined in terms of curvature and the singular structure of α and which represents the characteristic class $\phi(F)$. Moreover, the theory constructs a canonical and functorial transgression current $T = T(\phi, \alpha)$ with the property that

$$\phi(\Omega_F) - \phi(\Omega_{F,\alpha}) = dT.$$

When $\mathrm{rank}(E) = \mathrm{rank}(F)$, this often yields an equation of the form

$$\phi(\Omega_F) - \phi(\Omega_E) = \mathrm{Res}_\phi[\Sigma] + dT$$

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where $[\Sigma]$ is the current given by integration over the singular set $\Sigma \stackrel{\text{def}}{=} \{x \in X : \alpha_x \text{ is not an isomorphism}\}$ and where Res_ϕ is a smooth "residue" form which is expressed canonically in terms of the curvatures Ω_E and Ω_F of E and F. One can think of $S_\phi = \mathrm{Res}_\phi[\Sigma]$ as the characteristic current associated to $\alpha : E \longrightarrow F$ which represents the class $\phi(F) - \phi(E) \in H^*_{\text{de Rham}}(X)$.

When $\mathrm{rank}(E) > \mathrm{rank}(F)$, there are analogous formulas:

$$\phi(\Omega_E) - \phi(\Omega_0) = \mathrm{Res}_\phi[\Sigma] + dT$$

where $\phi(\Omega_0)$ is a closed differential form with $L^1_{\text{loc}}(X)$-coefficients whose cohomology class is more subtly determined by the global geometry.

2. An example: Complex line bundles

Suppose that E and F are complex line bundles over an oriented manifold X and that $\alpha : E \longrightarrow F$ is a smooth bundle map. If α vanishes nondegenerately, we define its divisor to be the current $\mathrm{Div}(\alpha) = [\Sigma]$ associated to the oriented codimension-2 submanifold $\Sigma = \{x \in X : \alpha_x = 0\}$. Set

$$f = \tfrac{i}{2\pi}\Omega_F \quad \text{and} \quad e = \tfrac{i}{2\pi}\Omega_E$$

where Ω_E and Ω_F are the curvature 2-forms of the connections. Then it is shown that there exists a 1-form σ on X with $L^1_{\text{loc}}(X)$-coefficients such that

$$f - e = \mathrm{Div}(\alpha) + \tfrac{i}{2\pi} d\sigma, \tag{1}$$

where σ is defined by $D\alpha = \sigma\alpha$. Passing to cohomology, we capture the first Chern class $c_1(F) - c_1(E) = [f] - [e] = [\mathrm{Div}(\alpha)] \in H^2_{\text{de Rham}}(X)$. Note that when E is trivial, equation (1) represents a C^∞ enhancement of the classical Poincaré-Lelong Formula for sections of the bundle F.

More precisely, formula (1) is shown to hold for any bundle map α which is *atomic*, that is, for which

$$\frac{da}{a} \in L^1_{\text{loc}}(X),$$

where a is the complex-valued function which represents α in the given local frames. This condition is independent of the choice of frames, as is the closed current $\mathrm{Div}(\alpha) = d\left(\frac{1}{2\pi i}\frac{da}{a}\right)$ (cf. [HS]). Note that locally $\sigma = \frac{da}{a} + \omega_F - \omega_E$. The following general result is formula (1) when $\phi(u) = u$.

Theorem A. *Fix a polynomial $\phi(u) \in \mathbf{C}[u]$ in one indeterminate. Then for any atomic bundle map $\alpha : E \longrightarrow F$ as above, there exists a differential form T with L^1_{loc}-coefficients on X such that*

$$\phi(f) - \phi(e) = \left\{\frac{\phi(f) - \phi(e)}{f - e}\right\} \mathrm{Div}(\alpha) + dT \tag{2}$$

where $\frac{\phi(f)-\phi(e)}{f-e}$ is the obvious polynomial in e and f. In fact, the $L^1_{\text{loc}}(X)$-form T is given explicitly by

$$T = \tfrac{i}{2\pi}\left\{\frac{\phi(f) - \phi(e)}{f - e}\right\}\sigma.$$

Combining this result with the kernel-calculus of Harvey and Polking [HP] yields a new proof of the Riemann-Roch Theorem for vector bundles over algebraic curves.

3. The general procedure; approximation modes

Let E and F be smooth vector bundles with connections D_E and D_F over a manifold X where, for simplicity in the following discussion, we assume $\operatorname{rank}(E) \leq \operatorname{rank}(F)$. Now for bundle maps

$$\alpha : E \longrightarrow F \quad \text{and} \quad \beta : F \longrightarrow E$$

these connections can be transplanted via the formulas

$$\overleftarrow{D}^{\alpha,\beta} = \beta \circ D_F \circ \alpha + (1 - \beta\alpha) \circ D_E \quad \text{and} \quad \overrightarrow{D}^{\alpha,\beta} = \alpha \circ D_E \circ \beta + D_F \circ (1 - \alpha\beta)$$

to define induced connections on E and F respectively. When E and F are isomorphic and $\beta = \alpha^{-1}$, one recovers the standard gauge transformations. If α is injective, one can introduce metrics on E and F and define

$$\beta = (\alpha^* \alpha)^{-1} \alpha^*,$$

yielding a *pull back connection* $\overleftarrow{D}$ and a *push forward connection* $\overrightarrow{D}$. In general this procedure breaks down on the *singular set* $\Sigma \equiv \{x : \alpha_x \text{ is not injective}\}$, since β becomes singular on Σ. To remedy this, we choose an *approximation mode* by fixing a C^∞-function $\chi : [0, \infty] \longrightarrow [0, 1]$ with $\chi' \geq 0$, $\chi(0) = 0$, and $\chi(\infty) = 1$. We then define a smooth approximation β_s to β by

$$\beta_s = \chi\left(\frac{\alpha^* \alpha}{s}\right)\beta \quad \text{for } s > 0.$$

Plugging β_s into the formulas above gives a family of smooth connections $\overrightarrow{D}_s$ on F (and $\overleftarrow{D}_s$ on E). As $s \to 0$, $\beta_s \to \beta$ uniformly on compacta in $X - \Sigma$. If $\chi(t) \equiv 1$ for $t \geq 1$, then $\beta_s = \beta$ outside the neighborhood $\{x \in X : \|\alpha_x\|^2 < s\}$ of Σ. Such a choice of χ is called a *compactly supported approximation mode.* Another important case is the *algebraic approximation mode* where $\chi(t) = t/(1+t)$. Here the family of connections $\overrightarrow{D}_s$ is directly related to the *Grassmann graph construction* of R. MacPherson. When working in this mode with the tautological bundle map $\alpha : \pi^* E \longrightarrow \pi^* F$ over the total space of $\pi : \operatorname{Hom}(E, F) \to X$, the family $\overrightarrow{D}_s$ extends smoothly to a fibrewise compactifiation of $\operatorname{Hom}(E, F)$. This has strong consequences for characteristic forms in the curvature of $\overrightarrow{D}_s$.

Let $\overrightarrow{\Omega}_s$ denote the curvature of $\overrightarrow{D}_s$, and fix an Ad-invariant polynomial ϕ on the Lie algebra of the structure group of F.

Definition A. Suppose that the limit

$$\phi((\overrightarrow{D})) \stackrel{\text{def}}{=} \lim_{s \to 0} \phi(\overrightarrow{\Omega}_s)$$

exists as a current on X. Then $\phi((\overrightarrow{D}))$ is called the *ϕ-characteristic current associated to the singular push forward connection $\overrightarrow{D}$ on F.* An analogous definition holds for the singular pull back connection $\overleftarrow{D}$ on E.

Note that $\phi((\overrightarrow{D}))$ is automatically d-closed and represents the ϕ-characteristic class of F. Over the subset $X - \Sigma$, $\overrightarrow{D}$ is a smooth connection, and $\phi((\overrightarrow{D}))$ is a smooth differential form which we denote by $\phi(\overrightarrow{\Omega}_0)$. Note that when $\operatorname{rank}(E) = \operatorname{rank}(F)$, $\overrightarrow{D}$ is gauge equivalent via α to D_E over $X - \Sigma$, and therefore $\phi(\overrightarrow{\Omega}_0) = \phi(\Omega_E)$ extends smoothly across the singular set Σ. When $\operatorname{rank}(E) < \operatorname{rank}(F)$, [HL1] establishes conditions on α which guarantee that $\phi(\overrightarrow{\Omega}_0)$ extends across Σ as a d-closed L^1_{loc}-form. This gives a decomposition

$$\phi((\overrightarrow{D})) = \phi(\overrightarrow{\Omega}_0) + S_\phi \tag{3}$$

where $\phi(\overrightarrow{\Omega}_0)$ denotes the $L^1_{\text{loc}}(X)$-extension and where S_ϕ is a current on X with the property that

$$dS_\phi = 0 \quad \text{and} \quad \operatorname{supp}(S_\phi) \subseteq \Sigma.$$

Each term in (3) represents a de Rham cohomology class on X which can be nonzero even when E and F are trivial bundles.

The detailed structure of S_ϕ and its independence of approximation mode is established by considering a family of transgression forms T_s with

$$dT_s = \phi(\Omega_F) - \phi(\overrightarrow{\Omega}_s)$$

and finding conditions on α so that $T \overset{\text{def}}{=} \lim_{s \to 0} T_s$ exists in $L^1_{\text{loc}}(X)$. The *transgression* T satisfies

$$dT = \phi(\Omega_F) - \phi(\overrightarrow{\Omega}_0) - S_\phi$$

and is functorial under appropriately transverse maps between manifolds. In many cases S_ϕ can be written as

$$S_\phi = \operatorname{Res}_\phi[\Sigma]$$

where Res_ϕ is a smooth form on X, expressed as a universal Ad-invariant polynomial in Ω_E and Ω_F. In particular Res_ϕ is completely determined by computing its associated cohomology class in the universal setting.

Combining this gives a *Chern-Weil Theorem for bundle maps*:

$$\phi(\Omega_F) - \phi(\overrightarrow{\Omega}_0) = \operatorname{Res}_\phi[\Sigma] + dT \tag{4}$$

where $[\Sigma]$ is a current canonically determined by the singular structure of α. Special cases are discussed in the following sections.

4. Universal Thom classes

When E is the trivial line bundle, α becomes a cross-section of F.

Definition B. The cross-section α is said to be *atomic* if whenever α is written locally as $\alpha = \sum a_j e_j$ in terms of a local frame field e, the $\mathbf{R}^m$-valued function $a = (a_1, \dots, a_m)$ satisfies

$$\frac{da^I}{|a|^{|I|}} \in L^1_{\text{loc}}(X) \quad \text{for all } |I| < m.$$

Elementary criteria for atomicity are given in [HS]. Roughly speaking, any smooth section which vanishes "algebraically" on a set of the appropriate "Minkowski codimension" is atomic. For example, any real analytic section with zeros of codimension $\geq m$ is atomic.

It is proved in [HS] that the vanishing of an atomic section α determines a unique, d-closed current $\mathrm{Div}(\alpha)$ of real codimension-m, called the *divisor* of α. Locally, the divisor is defined by

$$\mathrm{Div}(\alpha) = d(a^*\theta)$$

where θ is the normalized solid angle kernel on $\mathbf{R}^m$. This current $\mathrm{Div}(\alpha)$ is integrally flat, and in particular, when its mass is finite, it is a rectifiable cycle in the sense of Federer [F].

Suppose that F is either complex, or real and oriented, and that ϕ is the top Chern polynomial $\widetilde{\det}(A) = \det(\frac{i}{2\pi}A)$ or the Euler polynomial $\widetilde{\mathrm{Pf}}(A) = \mathrm{Pfaff}\left(-\frac{1}{2\pi}A\right)$ respectively. (If $\dim_{\mathbf{R}} F$ is odd, then $\phi \equiv 0$.) When α is atomic, the theory produces a canonical $L^1_{\mathrm{loc}}(X)$-form σ on X called the *spherical potential* such that

$$\phi(\Omega_F) - \mathrm{Div}(\alpha) = d\sigma.$$

Furthermore, for each approximation mode there is a smooth family of connections $\overleftarrow{D}_s$, $0 < s \leq \infty$ on F, so that $\tau_s \equiv \phi(\overleftarrow{\Omega}_s)$ satisfies

$$\tau_\infty = \phi(\Omega_F) \quad \text{and} \quad \lim_{s\to 0} \tau_s = \mathrm{Div}(\alpha).$$

There is also a family of L^1_{loc}-forms σ_s with $\lim_{s\to 0}\sigma_s = \sigma$ in $L^1_{\mathrm{loc}}(X)$ and

$$\tau_s - \mathrm{Div}(\alpha) = d(\sigma - \sigma_s).$$

It is useful to view this construction on the total space of the bundle $\pi : F \longrightarrow X$. Consider the pull back $\mathbf{F} = \pi^* F$ with the pull back connection. Over F there is a *tautological cross section* $\boldsymbol{\alpha}$ of $\mathbf{F}$ given by $\boldsymbol{\alpha}(v) = v$. This section is atomic, and the theory applies. Each approximation mode χ gives a smooth family of closed differential forms $\boldsymbol{\tau}_{\mathbf{s}}$, $0 < s \leq \infty$ on F, which are expressed canonically in terms of the connection, and which represent the *Thom class* of F. For example, when F is real of dimension $2n$ and $\chi(t) = 1 - 1/\sqrt{1+t}$, $\boldsymbol{\tau}_{\mathbf{s}}$ is given by the formula

$$\boldsymbol{\tau}_{\mathbf{s}} = \frac{s}{\sqrt{|u|^2 + s^2}}\widetilde{\mathrm{Pf}}\left(\frac{Du^t Du}{|u|^2+s^2} - \Omega_F\right)$$

where u represents the tautological section and Du is its covariant derivative.

Theorem B. *For any approximation mode the form $\boldsymbol{\tau}_{\mathbf{s}}$ is a closed $2n$-form on F which dies at infinity, is integrable on the fibres with integral one, restricts to be $\widetilde{\mathrm{Pf}}(\Omega_F)$ on X, converges as $s \to 0$ to the current $[X]$ represented by the zero-section, and converges as $s \to \infty$ to $\widetilde{\mathrm{Pf}}(\Omega_F)$.*

The *Thom form* $\boldsymbol{\tau}_{\mathbf{s}}$ can be written as the Chern-Weil image of a universal form in the equivariant de Rham complex of $\mathbf{R}^m$. In each approximation mode there is an explicit universal formula for the Thom forms $\boldsymbol{\tau}_{\mathbf{s}}$. If $\chi(t) \equiv 1$ for $t \geq 1$, we have $\mathrm{supp}(\boldsymbol{\tau}_{\mathbf{s}}) \subset \{v \in F : |v| \leq s\}$. In particular, $\boldsymbol{\tau}_{\mathbf{s}}$ has *compact support in each fibre.*

There are analogous Thom forms associated to the top Chern form when F is complex.

5. Rectifiable Grothendieck-Riemann-Roch

Another basic formula arises from the theory when considering an atomic section $\alpha \in \Gamma(V)$ of an even-dimensional, real vector bundle $V \longrightarrow X$ with spin structure. Clifford multiplication by α determines a bundle map $\alpha : \mathcal{S}^+ \longrightarrow \mathcal{S}^-$ between the positive and negative complex spinor bundles of V. Consider the function on matrices $\mathbf{ch}(A) \stackrel{\text{def}}{=} \operatorname{trace}\{\exp(\frac{1}{2\pi i}A)\}$ which gives the Chern character. Suppose $\mathcal{S}^+$ and $\mathcal{S}^-$ carry connections induced from a metric connection on V, and let $\Omega_{\mathcal{S}^\pm}$, Ω_V denote the curvature matrices of these connections.

Theorem C. *The following equation of forms and currents holds on X:*

$$\mathbf{ch}(\Omega_{\mathcal{S}^+}) - \mathbf{ch}\left(\Omega_{\mathcal{S}^-}\right) = \widehat{\mathbf{A}}(\Omega_V)^{-1}\operatorname{Div}(\alpha) + dT \tag{5}$$

where

$$\widehat{\mathbf{A}}(\Omega_V)^{-1} = \det{}^{\frac{1}{2}}\left\{\frac{\sinh(\frac{1}{4\pi}\Omega_V)}{\frac{1}{4\pi}\Omega_V}\right\}$$

is the series of differential forms on X which canonically represent, via Chern-Weil Theory, the inverse $\widehat{\mathbf{A}}$-class of V, and $T = \widehat{\mathbf{A}}^{-1}(\Omega_V)\sigma$ where σ is the spherical potential. More generally, for any complex bundle E with connection over X and $T = \operatorname{ch}(\Omega_E)\widehat{\mathbf{A}}^{-1}(\Omega_V)\sigma$ one has:

$$\mathbf{ch}(\Omega_{\mathcal{S}^+\otimes E}) - \mathbf{ch}\left(\Omega_{\mathcal{S}^-\otimes E}\right) = \mathbf{ch}(\Omega_E)\widehat{\mathbf{A}}(\Omega_V)^{-1}\operatorname{Div}(\alpha) + dT. \tag{6}$$

Equation (6) raises the Differentiable Riemann-Roch Theorem for embeddings [AH] to *the level of differential forms* and extends it to oriented subcomplexes which arise as divisors of some cross section of a bundle. One also obtains approximating families as with the Thom forms above.

There are formulas analogous to (6) when V is complex or Spin^c. The complex case gives a canonical version of a classical Grothendieck Theorem at the level of forms and currents. To be specific, let $j : Y \hookrightarrow X$ be a proper almost complex embedding of almost complex manifolds with normal bundle N. If the tangent bundles TY and TX are given connections compatible with the complex structures, then one has the following equation of forms and currents on X:

$$\begin{aligned}\left\{\mathbf{ch}(\Omega_{(\Lambda^{\text{even}}N^*)\otimes E}) - \mathbf{ch}\left(\Omega_{(\Lambda^{\text{odd}}N^*)\otimes E}\right)\right\} \wedge \mathbf{Todd}(\Omega_{TX}) \\ = \mathbf{ch}(\Omega_E) \wedge \mathbf{Todd}(\Omega_{TY})[Y] + dT\end{aligned}$$

for any vector bundle with connection E over Y. By passing to cohomology, this formula yields the commutativity of the diagram

$$\begin{array}{ccc} K(Y) & \xrightarrow{j_!} & K(X) \\ {\scriptstyle \mathbf{ch}(\,\cdot\,)\wedge\mathbf{Todd}(Y)}\downarrow & & \downarrow{\scriptstyle \mathbf{ch}(\,\cdot\,)\wedge\mathbf{Todd}(X)} \\ H^*(Y) & \xrightarrow[j_!]{} & H^*(X) \end{array}$$

where the $j_!$ represent the Gysin "wrong way" maps in K-theory and cohomology.

In these special Clifford multiplication cases this formalism has some similarities with Quillen's calculus of superconnections [Q] as developed in [MQ, BV, BGS*] and elsewhere. However, even in these cases there are substantial differences. We are concerned with transgressions and convergence questions under the weak atomic hypothesis and with the explicit structure of the limiting currents. Our theory also allows for a quite general choice of approximation mode. Choosing $\chi(t) = 1 - e^t$ puts us closest to Quillen's theory. However, choosing χ with $\chi(t) \equiv 1$ for $t \geq 1$ gives approximations supported in small neighborhoods of the singular set. Approximations where $\chi(t) = t/(1+t)$ admit nice compactifications (see [Z1]) and are related to MacPherson's Grassmann graph construction.

6. Quaternionic line bundles

Let $\alpha : E \longrightarrow F$ be an endomorphism of smooth quaternionic line bundles. The classifying space for quaternionic line bundles is the infinite quaternionic projective space $\mathbf{P}^\infty(\mathbf{H})$ whose cohomology is a polynomial ring on one generator $u \in H^2(\mathbf{P}^\infty(\mathbf{H}); \mathbf{Z})$ called the *instanton class.*

Theorem D. *Suppose E and F are provided with connections which are compatible with the quaternionic structure, and assume that α is atomic. Then for each $\phi \in \mathbf{R}[u]$ there exists a canonical $L^1_{\mathrm{loc}}(X)$-form T with the property that*

$$\phi(f) - \phi(e) = \left\{ \frac{\phi(f) - \phi(e)}{f - e} \right\} \mathrm{Div}(\alpha) + dT$$

where

$$f = \tfrac{1}{16\pi^2} tr\left\{\Omega_F^2\right\} \quad \textit{and} \quad e = \tfrac{1}{16\pi^2} tr\left\{\Omega_E^2\right\}$$

are the canonical representatives of the instanton class of E and F.

7. Degeneracy currents

For each bundle map $\alpha : E \longrightarrow F$ and integer k with $k < m = \mathrm{rank}(E) \leq n = \mathrm{rank}(F)$, we define a current which measures the locus where $\mathrm{rank}\,(\alpha) \leq k$. Specifically, let $\xi : G_r(E) \longrightarrow X$ be the Grassmann bundle of r-planes in E with $r = m - k$, and let $U \longrightarrow G_r(E)$ be the tautological r-plane bundle. Note that U is a subbundle of ξ^*E and that α pulls back to a bundle map $\xi^*\alpha : \xi^*E \to \xi^*F$. We say that the bundle map α is *k-atomic* if the restriction $\hat{\alpha} = \xi^*\alpha|_U$ is an atomic section of $\mathrm{Hom}(U, \xi^*F)$ over $G_r(E)$. Under this hypothesis we define the kth *degeneracy current* of α to be the current push-forward

$$\mathbb{D}_k(\alpha) = \xi_* \mathrm{Div}(\hat{\alpha}).$$

When α is algebraic, $\mathbb{D}_k(\alpha)$ is a cycle whose class in the Chow ring $A^*(X)$ is the one defined by Fulton in [Fu]. For given connections on E and F there are canonical families of smooth forms TP_s and L^1_{loc}-forms S_s on X, for $0 < s \leq \infty$, such that

$$\mathbb{D}_k(\alpha) = \mathrm{TP}_s + dS_s \tag{7}$$

where $\lim_{s\to 0} S_s = 0$ in L^1_{loc} and where in the complex case

$$\mathrm{TP}_\infty = \det_{m\times m}((c(E-F)_{n-i+j})).$$

Here $c(E-F) \overset{\mathrm{def}}{=} 1 + c_1 + c_2 + \cdots = c(\Omega_F)/c(\Omega_E)$ in the ring of even forms on X. There is an analogous formula in the real case. These *Thom-Porteous families* TP_s are established in [HL2].

8. Geometric formulas

The results above give a wide variety of formulas relating characteristic forms to singularities of maps. For example, consider $\alpha : \underline{\mathbf{C}}^{k+1} \to F$ corresponding to $(k+1)$-sections $s_0, \dots, s_k$ of a complex bundle with connection F. If α is k-atomic, the top degeneracy current $\mathbb{D}_k(\alpha) \overset{\mathrm{def}}{=} \mathbb{LD}(s_0, \dots, s_k)$ is defined and measures the locus of linear dependence of these sections. From (7) we have

$$c_{n-k}(\Omega_F) - \mathbb{LD}(s_0, \dots, s_k) = dT$$

where T is a canonical L^1_{loc}-form on X. There are corresponding formulas for higher-order degeneracies.

A fundamental case occurs when $\alpha = df : TX \longrightarrow f^*TY$ is the differential of a smooth map $f : X \longrightarrow Y$ between manifolds. This yields, for example, a C^∞-version of the formula for the global Milnor current (cf. [Fu, 14.1.5]). When X and Y are oriented 4-manifolds, it yields a formula relating $f^*p_1(Y) - p_1(X)$ to a weighted sum of isolated "singular points" where rank$(df) = 2$ (cf. [M2] and [R]). There are also formulas of type (7) explicitly relating Pontrjagin forms to the singularities of projections $X^n \subset \mathbf{R}^{n+n'} \longrightarrow \mathbf{R}^k$.

Applications to CR-geometry include the following. Consider a generic smooth map $f : M \to \mathbf{C}^{k+1}$ of an oriented riemannian n-manifold M where $n - k = 2\ell > 0$. Then there are formulas

$$p_\ell(\Omega_M) = (-1)^\ell \mathbb{C}r(f) + dT$$

where $p_\ell(\Omega_M)$ is the ℓth Pontrjagin form of M and where $\mathbb{C}r(f)$ is a current associated to the *CR critical set*: $\{x \in M : df : TM \otimes_{\mathbf{R}} \mathbf{C} \longrightarrow \mathbf{C}^{k+1}$ is not surjective$\}$. For example, if $\dim M = 4$ and $f : M \longrightarrow \mathbf{C}^3$ is an immersion, then we have the formula

$$p_1(\Omega_M) = -\mathbb{C}r(f) + dT$$

where $\mathbb{C}r(f)$ is the (generically finite) set of complex tangencies to $f(M) \subset \mathbf{C}^3$ taken with appropriate indices. More complicated formulas involving higher-order complex tangencies and Shur functions are derived in [HL2].

Another application associates geometric currents to pairs of complex structures. Suppose J_1 and J_2 are smooth almost complex structures on a vector bundle $E \longrightarrow X$, and let $E \otimes \mathbf{C} = E_1 \oplus \overline{E}_1 = E_2 \oplus \overline{E}_2$ be the associated splittings. Restriction and projection give a complex bundle map $p : E_1 \longrightarrow E_2$ to which there are associated degeneracy currents and geometric formulas. For example, let $n = \mathrm{rank}(E)$ and suppose $\lambda \overset{\mathrm{def}}{=} \Lambda^n_{\mathbf{C}} p$ is atomic. Then we have the characteristic current

$$\mathbb{C}r(J_1, J_2) = \mathrm{Div}(\lambda)$$

which is supported in the set $\{x \in X : \ker(J_1 + J_2) \neq \{0\}\}$. Let D_k be a connection on E such that $D_k(J_k) = 0$, and set $e_k = c_1(D_k)$ for $k = 1, 2$. Then for any $\phi(t) \in \mathbf{C}[t]$, there is an L^1_{loc} form σ_ϕ on X such that

$$\phi(e_2) - \phi(e_1) = \frac{\phi(e_2) - \phi(e_1)}{e_2 - e_1} \mathbb{C}r(J_1, J_2) + d\sigma_\phi.$$

Similar formulas, which involve the higher degeneracy currents, are derived in [HL2].

The theory produces formulas of the above type in the theory of foliations. It is also possible, using this theory, to rederive and generalize formulas of Sid Webster [W*]. It should be remarked that discussions with Jon Wolfson (cf. [Wo] and [MW]) about Webster's formulas served as an inspiration for this work.

Each geometric formula discussed in [HL2] carries with it a smooth 1-parameter family of analogous formulas coming from $\vec{D}_s$. As $s \to 0$, the characteristic forms converge to the degeneracy current.

In [HL3] the authors study residue formulas associated to bundle maps of general rank. These formulas simultaneously involve several of the degeneracy strata of the map, each paired with its own residue form. They promote the work of R. MacPherson [M1, M2] to the level of characteristic forms and currents.

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