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ALEXANDER'S AND MARKOV'S THEOREMS IN DIMENSION FOUR

SEIICHI KAMADA

ABSTRACT. Alexander's and Markov's theorems state that any link type in R^3 is represented by a closed braid and that such representations are related by some elementary operations called Markov moves. We generalize the notion of a braid to that in 4-dimensional space and establish an analogue of these theorems.

1. INTRODUCTION

Any link type in R^3 is represented by a closed braid, and two closed braids represent the same link type if and only if they are related by braid isotopies, stabilizations, and their inverse operations. These facts are well known as Alexander's theorem and Markov's theorem (cf. [B; A; Mo]). The latter is restated as follows: Two braids have ambient isotopic closures in R^3 if and only if they are related by conjugations, stabilizations, and their inverse operations (Markov moves). These theorems enable braid theory to play an important role in (classical) knot theory (for example, [J]). We have a natural analogue of them in 4-dimensional space.

There seem to be many generalizations of the notion of classical braids to higher dimensions ([D; G1; G2; BS; MS], etc.). We use, as a generalization of a classical braid, the notion of a 2-dimensional braid due to O. Viro (cf. [K2]). A similar notion was studied by L. Rudolph as a braided surface [R1; R2].

Definition. A *2-dimensional braid* (of degree m) is a compact oriented surface B embedded in bidisk $D_1^2 \times D_2^2$ such that (1) the restriction of the second factor projection $p_2 : D_1^2 \times D_2^2 \to D_2^2$ to B is an oriented branched covering map of degree m and (2) the boundary of B is a trivial closed braid $X_m \times \partial D_2^2 \subset D_1^2 \times \partial D_2^2 \subset \partial(D_1^2 \times D_2^2)$, where X_m is fixed m points in the interior of D_1^2.

We do not require that the associated branched covering map $B \to D_2^2$ is *simple*, although it is assumed in [K2; R1; R2].

Two 2-dimensional braids are *equivalent* if they are ambient isotopic by a fiber-preserving isotopy of $D_1^2 \times D_2^2$, where we regard $D_1^2 \times D_2^2$ as a D_1^2-bundle over D_2^2, keeping $D_1^2 \times \partial D_2^2$ fixed. We do not distinguish equivalent 2-dimensional braids. Two 2-dimensional braids are *braid isotopic* if we can deform one to the other by an isotopy of $D_1^2 \times D_2^2$ keeping the condition of a 2-dimensional braid and $D_1^2 \times \partial D_2^2$ fixed.

Let B and B' be 2-dimensional braids of the same degree m in $D_1^2 \times D_2^2$ and in $D_1^2 \times D_2^{2\prime}$. Take a boundary connected sum $D_2^2 \natural D_2^{2\prime}$ of D_2^2 and $D_2^{2\prime}$

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which is also a 2-disk. Then $B \cup B'$ in $D_1^2 \times (D_2^2 \natural D_2^{2'})$ forms a 2-dimensional braid. We call it a *braid sum* of B and B' and denote it by $B \cdot B'$, which is uniquely determined up to equivalence. The family of equivalence classes of 2-dimensional braids of degree m, together with this braid sum operation, is a commutative semi-group. We call it the *2-dimensional braid semi-group* of degree m.

A 2-dimensional braid B is naturally extended to a closed oriented surface $\widehat{B}$ embedded in $D_1^2 \times S^2 = D_1^2 \times (D_2^2 \cup \overline{D_2^2})$ with $\widehat{B} \cap D_1^2 \times \overline{D_2^2} = X_m \times \overline{D_2^2}$. Identifying $D_1^2 \times S^2$ with the tubular neighborhood of the standard 2-sphere about the t-axis in R^4, we assume $\widehat{B}$ to be a surface embedded in R^4 and call it the *closure* of B (about the t-axis) in R^4 or a *closed 2-dimensional braid*.

Theorem 1 (Viro [K1]). *Any closed oriented surface embedded in R^4 is ambient isotopic to a closed 2-dimensional braid in R^4.*

In [K1] a closed 2-dimensional braid in R^4 is treated as a sequence of closed braids in classical dimension, similar ideas to which are found in [G; R1; K3].

Two operations, "conjugations" and "stabilizations", are defined for 2-dimensional braids as a natural analogue of classical ones as follows:

Regard D_1^2 as $I_1^1 \times I_2^1$ and X_m as the set $\{z_1, \cdots, z_m\} \subset D_1^2$ where I_i^1 $(i = 1, 2)$ is the interval $[-1, 1]$ and z_i $(i = 1, \cdots, m)$ is point $(0, 1-2^{-i})$.

Let b be a classical braid of degree m in $D_1^2 \times I$ with $\partial b = b \cap D_1^2 \times \partial I = X_m \times \partial I$. The product $b \times S^1$ is a collection of annuli embedded in $D_1^2 \times I \times S^1$. Identify $I \times S^1$ with the collar neighborhood $N(\partial D_2^2)$ of ∂D_2^2 in D_2^2 so that $\partial D_2^2 \times \{1\} = \partial D_2^2$ and $D_1^2 \times I \times S^1$ with $D_1^2 \times N(\partial D_2^2)$.

Up to equivalence, a 2-dimensional braid B is assumed to satisfy that $B \cap D_1^2 \times N(\partial D_2^2)$ is the product $X_m \times N(\partial D_2^2)$. Replace it by $b \times S^1$ and we have a 2-dimensional braid. It is called a 2-dimensional braid obtained from B by the *conjugation* associated with b or the *conjugate* of B by b.

For a nonnegative integer a, let T_a be a homeomorphism of $D_1^2 = I_1^1 \times I_2^1$ such that $T_a(x, y_i) = (x, y_{i+a})$ for each $x \in I_1^1$ and $y_i = 1 - 2^{-i} \in I_2^1$ $(i = 0, 1, 2, \cdots)$, and let $\widetilde{T_a}$ be the homeomorphism of $D_1^2 \times D_2^2$ which acts on the first factor by T_a and on the second factor by the identity. For a 2-dimensional braid B of degree m, $\widetilde{T_a}(B)$ satisfies condition (1) of the definition of a 2-dimensional braid and condition (2) replaced X_m by $T_a(X_m) = \{z_{a+1}, \cdots, z_{a+m}\}$. Up to equivalence, we may assume that $\widetilde{T_a}(B)$ is contained in $I_1^1 \times (y_a, y_{m+a+1}) \times D_2^2 \subset I_1^1 \times I_2^1 \times D_2^2 = D_1^2 \times D_2^2$. For nonnegative integers a and b, let $\iota_a^b(B)$ be the union $\{z_1, \cdots, z_a\} \times D_2^2 \cup \widetilde{T_a}(B) \cup \{z_{m+a+1}, \cdots, z_{m+a+b}\} \times D_2^2$, which is a 2-dimensional braid of degree $m+a+b$. ι_a^b is an injection from the 2-dimensional braid semi-group of degree m into that of degree $m + a + b$.

By [K2] the equivalence class of a 2-dimensional braid of degree 2 is determined only by the number of branch points. Let B^* be a unique, up to equivalence, 2-dimensional braid of degree 2 with two branch points. It is a unique generator of the 2-dimensional braid semi-group of degree 2. A 2-dimensional braid B' of degree $m+1$ is said to be obtained from a 2-dimensional braid B of degree m by a *stabilization* if B' is equivalent to the braid sum $\iota_0^1(B) \cdot \iota_1^0(B^*)$.

Theorem 2. *Two 2-dimensional braids have ambient isotopic closures in R^4 if and only if they are related by braid isotopies, conjugations, stabilizations, and their inverse operations.*

Remark. A *simple* 2-dimensional braid is a 2-dimensional braid B whose associated branched covering $B \to D_2^2$ is simple. An advanced version of Theorem 1 is found in [K1]: Any closed oriented surface embedded in R^4 is ambient isotopic to the closure of a *simple* 2-dimensional braid. Then we have a natural question which is open at present: For two simple 2-dimensional braids with ambient isotopic closures in R^4, are they related only through *simple* 2-dimensional braids by braid isotopies, conjugations, stabilizations, and their inverse operations?

We work in the piecewise linear category, and surfaces in 4-space are assumed to be embedded properly and locally flatly.

2. Outline of the proof

Our proof follows [B]. Let ℓ denote the t-axis of R^4 and $\pi : R^4 \to R^3$ the projection along ℓ. We say that an oriented 2-simplex A in R^4 is *in general position with respect to* ℓ if there is no 3-plane in R^4 containing both A and ℓ. Then $\pi(A)$ is an oriented 2-simplex in R^3 which forms, together with the origin of R^3, an oriented 3-simplex. Using the orientation of R^3, we define the *sign* of A valued in $\{\pm 1\}$. Theorem 1 is shown as follows: Let F be a closed oriented surface in R^4 and K a *division* of F, which is a certain kind of tessellation by 2-simplices (not a triangulation but its generalization). We may assume that each 2-simplex of K is in general position with respect to ℓ. The number of 2-simplices of K with negative sign is denoted by $h(K)$. If $h(K) = 0$, then F is a closed 2-dimensional braid about ℓ in R^4.

Lemma 1 (Sawtooth Lemma). *If there is a 2-simplex A of K with negative sign, then there is a sawtooth over A of F.*

A *sawtooth* is a family of 3-simplices in R^4 satisfying a nice condition such that the surgery result of K is a division K' of another surface in R^4 with $h(K') = h(K) - 1$. By this lemma, we can deform F to a closed 2-dimensional braid.

As an analogue of [B], we can define some *operations* which transform a division of a surface in R^4 without changing the isotopy type of the surface. A *deformation chain* is a sequence of divisions of surfaces in R^4 connected by operations.

Lemma 2. *Let K, K' be a deformation chain with $h(K) = h(K') \neq 0$. Then there is a deformation chain $K, K_1, \cdots, K_s, K'$ for some $s \geq 1$ such that $h(K_i) < h(K)$ for i $(i = 1, \cdots, s)$.*

Lemma 3. *Let K, K', K'' be a deformation chain with $h(K') > h(K)$ and $h(K') > h(K'')$. Then there is a deformation chain $K, K_1, \cdots, K_s, K''$ for some $s \geq 1$ such that $h(K_i) < h(K')$ for i $(i = 1, \cdots, s)$.*

By the argument in [B], we have Theorem 2 as a consequence of Lemmas 2 and 3. The details will appear elsewhere.

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DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY, SUMIYOSHI, OSAKA 558, JAPAN
E-mail address: `g54042@jpnkudpc.bitnet`