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Harmonic analysis, 2nd edition, by Henry Helson, Hindustan Book Agency and Helson Publishing Co., 1995, vii+227 pp., $24.00. No ISBN number.

The vast empire in the world of mathematics known as Fourier analysis or harmonic analysis has one of the best-recorded histories of all mathematical subject areas. This empire is replete with teeming cities, backwater hamlets, towering monuments, and perhaps a few slums. Ultimately based on the Pythagorean theorem, which has been traced in its full generality in Middle Eastern cuneiform tablets to an era more than a millennium before Pythagoras, trigonometry was developed by the Greeks of Euclid's time as the study of the relation between chords and arcs on a circle in order to construct models of stellar and planetary motion. If the astronomers of Ptolemy's time had been asked what they were doing with the mathematical models, they would probably have expressed the faith that a finite number of epicycles on top of epicycles could approximate the path of a heavenly body within the limits of observational error for all time (more observations would only mean more epicycles). This faith, seen from the perspective of 20th-century harmonic analysis, amounts to the belief that the path of a celestial body can be pictured as an almost-periodic function of time. The 2,000-year journey from faith to clear understanding is marked by many great events.

The clever replacement of the chord by a half-chord (sine[1]) seems to be due to early Hindu mathematicians such as Aryabhata (6th century). A definition of trigonometric functions with the potential for abstraction beyond the confines of circles and triangles is due to the 16th-century Hindu mathematician Jyesthadeva, who explicitly described a power series equivalent to the Maclaurin series of the arctangent. The Scot James Gregory derived this same series about a century later. A similar power-series expression for the square of an arc as a function of its height and the diameter of the circle was given in Japan by Takebe Kenko a few decades after Gregory.

Almost as soon as the calculus of trigonometric functions was understood, these functions began to find application in the description of natural phenomena. These simple and very concrete functions turned out to have a wealth of symmetry properties based on the fundamental one, their periodicity. From this seemingly coarse material ever more elaborate patterns have been woven. The result is an enormous tapestry full of beautiful pictures, many of which have in common only the fact that they appear on the same basic fabric. To find a common thread between some of them is a challenge. For example, to take a well-known example cited by Loomis, anything less than an intensive study of Wiener's Tauberian theorem as Wiener stated it and the theorem that every closed ideal of $L^1(\mathbb{R}^1)$ is contained in a maximal ideal would fail to reveal any connection between the two, yet the latter theorem is colloquially known as the Wiener Tauberian Theorem in the context of Banach algebras. (Of course what the original theorem had to do with

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[1] After passing through Arabic, Aryabhata's descriptive Sanskrit word for the sine, *jya*, meaning bowstring, came into Latin as the utterly undescriptive word *sinus*, meaning a cavity.

Tauberian-type results was far from obvious to the reviewer, even after reading Wiener's account of it.)

If harmonic analysts are ever allowed to construct their own calendar, its epoch—Day One—will be in the mid-18th century, when the famous problem of the vibrating string (the one-dimensional wave equation) was solved by Daniel Bernoulli using separation of variables and undetermined coefficients to superimpose the functions that we now recognize as the eigenfunctions so as to fulfill the initial conditions. This famous solution was followed by a famous controversy over the admissibility of such representations, leading ultimately to the clarification of what is meant by a function. Half a century later these functions again proved their worth in the classic study of heat conduction by Joseph Fourier, from whom the whole subject took its name.

As these episodes showed, the trigonometric functions were a gold mine, but getting the gold out involved the building of some very intricate machinery. The frustrating search for necessary and sufficient conditions for convergence of a Fourier series led to the proliferation of the wild counterexamples that still make real analysis the bane of graduate students. It was trigonometric series that provided the first examples of continuous, nowhere differentiable functions. Besides this rather negative contribution, the trigonometric functions have played a positive role in the growth of both real and complex analysis. Legendre, for example, recognized that the elliptic integrals that arose so frequently in elementary mechanical and geometric problems bore a strong resemblance to the integrals that define the inverse trigonometric functions, and he suggested that the inverses of these integrals would be simpler to study than the integrals themselves. He was right, of course, as Abel and Jacobi showed; both the double periodicity of the elliptic functions as functions of a complex variable and the theta functions that tame them grow from the same roots as the trigonometric functions.

As for real analysis, it is well-known that Georg Cantor was engaged in investigating the points where a trigonometric series would have to converge to zero if it converged to zero everywhere else when he came across the crucial idea of a point of accumulation; it was this concept that led him to the idea of an abstract point set (abstract at least by the standards of his time—his early sets were still sets of real numbers). Although Borel and Lebesgue first developed their theory of integration from other motives, Lebesgue was quick to spot its potential for bringing order and unity into the study of trigonometric series and integrals. Try, if you can, to imagine how we would explain these things to graduate students without being able to talk about convergence and summability for L^p-functions or the Lebesgue set of a function as a minimal set on which its Fourier series must be summable.

Meanwhile, other organizing principles in harmonic analysis were growing from other roots. The method of separation of variables can be applied in any coordinate system, and when the classical partial differential equations of mathematical physics are studied in cylindrical and spherical coordinates, they lead to expansions in Bessel functions and Legendre polynomials. The analogy with trigonometric series served as a guide to a general theory of orthogonal expansions. Comparison of these different kinds of expansions leads to further considerations of symmetry. The symmetry of the trigonometric/exponential functions came to be seen as one example of a more general type of symmetry. The exponential functions are the characters of the real numbers and the circle, but other groups have their own characters, and there are all kinds of groups. The theorem that the eigenfunctions for the classical Sturm-Liouville problems are complete is paralleled by the Peter-Weyl theorem, which asserts that the coordinates of a complete set of inequivalent irreducible representations of any compact group form a complete orthogonal set in L^2 of that group.

After the emergence of abstract harmonic analysis as the study of characters of topological groups, the distinction between classical and abstract came to have a clearer meaning in harmonic analysis than in any other branch of mathematics. A theorem is "classical", no matter when proved, if it concerns a specific type of expansion on Euclidean space or the torus. It is "abstract" if it refers to any other group (even Weyl's "classical" groups), to topological groups in general, or to explicit geometries and transformation groups of specific manifolds in higher dimension. That is about as far as any attempt to classify harmonic analysis can go without invoking the nomenclature of other disciplines such as Lie theory, group representations, and the like. (An excellent survey is the paper by K. I. Gross, "On the evolution of noncommutative harmonic analysis", *Amer. Math. Monthly* **85** (1974), 525–548.) A Euclid-style axiomatic treatment of the subject would either have so many axioms and definitions that only one who already knew the subject could keep them all straight, or else it would ignore large numbers of famous and important results. Each of the two halves of the empire is too large to be governed by a single emperor: there is no "fundamental theorem of harmonic analysis"[2], and, as Hewitt and Ross demonstrated a generation ago, a complete account of even the abstract half of the empire is a lengthy undertaking. Of course, the interconnections are an example of the common thread in mathematics that makes our discipline a powerful intellectual force.

The volume under review leans toward the classical half of the empire, to which the author himself has made significant contributions, especially in explicating the mysteries of the classical H^p spaces. The one chapter (Chapter 3) on general topological groups, however, is the best-written and most enjoyable introduction to this subject that the reviewer has seen. In the astonishingly short space of 30 pages the author gives a good account of the basic principles of abstract harmonic analysis, together with examples. The importance of positive-definiteness is stressed, and Bochner's theorem is used to prove the Kolmogorov extension theorem for measures on an infinite product of probability spaces.

Readers for whom the basic facts of both classical and abstract harmonic analysis are familiar territory may still find something new and interesting in the author's way of handling the material. His discussion of the Kronecker-Weyl theorem in the final chapter, in particular, puts the whole circle of ideas in good perspective. (This is not to say in the ultimate perspective, which would be difficult. Bochner once remarked to the reviewer that he had "never understood" this theorem—meaning, of course, that he had never worked out a way to make it a natural consequence of broad general principles.)

[2] In an attempt to get as far as possible with minimal tools a few years ago, the reviewer wrote some notes for students in which the Riemann-Lebesgue Lemma and the Plancherel Theorem were computed for step functions on R^n, then extended to L^2 "by contagion". The Gauss-Weierstrass, Abel-Poisson, and Bochner-Riesz kernels were computed by similar bare-fisted methods. Even so, to get from integrals to series required the Poisson summation formula and the well-known symmetry properties of the Fourier transform. Without at least five or six independent pillars, the edifice cannot even be begun.

The majority of the book consists of a very readable introduction to classical harmonic analysis (Chapters 1 and 2), in which the author attempts to derive as much as possible from the basic facts such as the Riemann-Lebesgue Lemma (which he calls *Mercer's Theorem*, perhaps on better grounds than one has for using the traditional name) and the Fejér and Poisson kernels. Certain nonstandard material is included here as well, including the theorem of the author and A. Beurling that a bounded-convolution measure on the line is a unit point mass (stated but not proved) and its analogue on the integers, due to J.-P. Kahane (stated and proved). Results of this type, asserting that any abstract entity possesssing a modicum of regularity must be a very familiar object, are what give classical harmonic analysis its charm for the reviewer. The subject is rich in them. For example, a theorem of R. Coifman asserts that a homeomorphism σ of Euclidean space such that the operation of composition with σ commutes with the Fourier transform is necessarily an orthogonal linear transformation.

The author is best-known to the reviewer as the creator of some of the most profound results in the classical Hardy spaces. This material is naturally well-developed here, including a nice derivation of Beurling's theorem characterizing the outer functions in H^2 from Szegö's formula for the distance from the constant function 1 to the trigonometric polynomials with constant term zero in the space $L^2(w(x)\,dx)$.

While the author did not intend to write a systematic exposition of the basic parts of harmonic analysis, the logic of the material selected in this volume entails developing a substantial number of basic results in order to make the book accessible to a reader with a modest grounding in general real and complex analysis. The author clearly loves these topics and has assembled a rich collection of little gems from the mine. For graduate students who wonder what the subject is about but are deterred by the length of the books of Zygmund, Bari, Hewitt and Ross, and others, this volume can be an excellent way to get one's feet wet and see whether it is worthwhile to take the plunge. The book has helpful and challenging exercises for the reader. Those who find the material enjoyable will then be ready for more systematic expositions in such books as Rudin's *Fourier analysis on groups*, Stein and Weiss' *Introduction to harmonic analysis on Euclidean spaces*, or Hoffman's *Banach spaces of analytic functions*. It should be emphasized that the great strength of the book lies in the large number of interesting special results it contains. Many of the prettiest facts of harmonic analysis are on display here in a very attractive setting.

It is precisely because of this wealth of particular results that harmonic analysis defies systematization. The author says very wisely in his preface, "This is not a treatise. If what follows is interesting and useful, no apology is offered for what is not here." No apology is needed, Prof. Helson, none at all.

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