BOOK REVIEWS

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Visual geometry and topology, by Anatoly Fomenko, Springer-Verlag, Berlin and Heidelberg, 1994, ix + 324 pp., $89.00, ISBN 3-540-53361-3, ISBN 0-387-53361-3

Anatoly Fomenko is a prolific mathematician and a prolific artist. He has published books on any number of topics in geometry and topology, all with his own illustrations, and he has completed hundreds of elaborate drawings, many based on mathematical themes. This book puts together some of his most engaging visual presentations of classical and modern geometry along with numerous full-page pen-and-ink creations. The effect is a book like none other. It is easy to recommend it to any professional mathematician. It would be nice to be able to recommend it to any graduate student or advanced undergraduate, the way many of us for years have been pushing the classic *Geometry and the imagination* (originally *Anschauliche Geometrie*) by Hilbert and Cohn-Vossen. Unfortunately that is not possible. The book has flaws that will keep it from becoming popular where it could do the most good. Many of these shortcomings could have been avoided, since most of them are due to a bad mathematical translation and a poor job of proofreading to go along with it. Perhaps the book will stand as a warning to those of us who occasionally find one of our own books being translated. It is simply impossible for a professional translator, no matter how skillful, to get things right unless he or she has some mathematical training. There is a solution to that problem—just have the translated manuscript checked by mathematicians who know the language and who work in approximately the same fields as the author of the book. That crucial step is missing here.

Several choices of words are simply not the ones that would be chosen by someone familiar with popular mathematical writing in English. We do not "wind" a strip to form a Möbius band; we "twist" it (p. 2). We do not speak of torus in the form of a "roll" (p. 42); rather we refer to a "doughnut" or a "bagel". We use the term "cross-cap" rather than "crossed cap" (p. 117) and "twisted handle" rather than "screwed handle" (p. 123). We say that a strip is "linked" by another one, not "hooked" by it (p. 125). We definitely use the term "osculating circle", not "adjoining circle" as on page 255, and "catenary" instead of "chain line" on page 259. Some of the instances of unusual terminology are worthy of Lewis Carroll. Consider the sentence on page 125: "Theorem 2. Any connected closed 2-manifold is homeomorphic either to a full cracknel of some genus..." Upon first reading

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that statement I assumed that a cracknel was some Russian equivalent of a pretzel, so the sentence retained its sense even though the term was totally unfamiliar. But no, it turns out that "cracknel" is in the *Oxford English Dictionary* with the definition: "A light, crisp kind of biscuit, of a curved or hollow shape." The last use of that term recorded in the OED was such an obscure expression. There are other cases where a technical term is used in a nontraditional way. Page 128 refers to "inversion" rather than "eversion" of a sphere, even though the term "eversion" is used in the introduction, almost certainly written directly by the author in English.

Translating from Russian to English presents some special challenges. While Russian has no articles, every noun in English is preceded either by an indefinite article, a definite article, or no article. A translator is always faced with one right choice and two wrong ones, and unfortunately in this book the choices are very often wrong. Most of the time it is merely annoying, but sometimes it is downright confusing. Consider on page 195 the sentence, "Recall that a function f on a manifold is called <u>the</u> integral of the vector field v if its derivative along this field is identically zero." The offending word is underlined—one of the most important points of Hamiltonian geometry is that a vector field can have more than one integral. Use of the indefinite article is required for clarity in this case.

There are very few places where there can be any fault with the drawings. One misleading example, however, is the illustration of the notion of an osculating circle (Figure 4.1.1). The osculating circle at a point of a plane curve in general separates a neighborhood of the point on the curve into two parts, one (locally) inside and one (locally) outside. It is of course possible for the curve to lie locally to one side of the osculating circle, as for example at a point of an ellipse where the curvature has a maximum or minimum, but this is not the case in the illustration. Figure 4.1.2 makes the same mistake for both lines of curvature on a hyperbolic paraboloid.

And what about the elaborate full-page pictures? Many readers are familiar with the author's imaginative pen-and-ink drawings from the collection published by the AMS a few years ago. In this book, however, frequently the connection between the pictures and the concepts being discussed seems artificial and unilluminating. As the author mentions, many of the drawings were commissioned to illustrate literary works rather than mathematical ideas. As a mathematician, he finds that geometric and topological ideas inspire and affect his designs, and it is natural to point out some of those influences. However, in many of these drawings, the connections with mathematical ideas in the book seem tenuous, and the captions are often vague. Even in the parts where one might expect to see some familiar patterns—for example, in the sections on fractals and on hyperbolic geometry—it is hard to relate the images to mathematical concepts.

There are a few lapses toward the beginning of the book that might drive away an unconfident reader. One of the most curious is on page 4, where it is stated, "It is not true that each graph can be realized as a set of points on a plane. For example, the graph in Fig. 1.1.6 (the set of edges of a tetrahedron) cannot be imbedded into a plane." The figure referred to shows a projection of the tetrahedron that has four extreme points so two edges must cross. Any reader following with a pencil and paper will discover that putting one vertex inside the triangle formed by the other three does indeed give an embedding of the set of edges of a tetrahedron. The author certainly meant to suggest the problem of embedding the complete graph on five vertices (the set of edges of a 4-simplex), and any mathematician would recognize that intent. But a beginner would be confused.

Sometimes the author appeals to authority in a place where it would be just as easy to give a plausible argument. On page 2, for example, he writes, "It can be verified that such a (Möbius) strip is already not homeomorphic to a flat ring (to a usual strip)." Why not just point out that the usual strip has two boundary curves while the Möbius strip has only one? It is also possible to give an elementary argument to show that nonorientable surfaces without boundary cannot be embedded in 3-space. First establish a lemma that a closed curve that meets any closed surface transversely must meet it in an even number of points. Then show that if an embedded surface included a Möbius strip, it would be possible to find a curve near the center of the strip meeting the strip (and the surface) in a single point, a contradiction. Such a lemma is plausible but subtle in the case of a topological or even a smooth embedding, but quite natural and direct for polyhedra. It might have been better to spend some more time on plausibility arguments for important topological and geometric facts rather than moving quickly into the formalism of simplicial topology or differential topology.

It seems strange that the author chooses to use a nonstandard notion of triangulation of a surface. By introducing a single interior vertex in a square with opposite sides identified, the author connects that central point to the same point on the boundary four times. If an edge is determined by its endpoints, this gives one edge adjacent to eight different triangles, in violation of one of the requirements. Rather we have the sentence on page 13, "As has already been mentioned above, we shall allow some simplexes to have two common vertices. This will not affect our calculations of homology groups and will simplify significantly (in some cases) the representation of topological spaces as unions of simplexes." That is true, but it is at variance with the combinatorial formalism introduced just a bit later. Although it is convenient, it is different from most other introductory treatments of triangulations, so it is important to warn the reader.

It is not clear why the material on visual hyperbolic geometry has been relegated to an appendix. It seems just as valid to include this material in the body of the volume, along with the treatments of visual symplectic topology and Hamiltonian geometry.

There is no index. That is a major fault. This omission severely limits the usability of the book, especially since one of the author's stated aims is to produce a book that a reader can pick up at any point. That only works when it is easy to find definitions of unfamiliar terms; without an index, that is close to impossible.

As it happens, it is no longer possible to purchase a copy of the first edition of this book. The publisher removed it from bookstore shelves soon after some of the above flaws were pointed out in other published reviews. Fortunately a new edition is currently being prepared, and professional mathematicians are collaborating with translators to avoid the problems that plague the original. Unfortunately the first edition will remain on library shelves for many years, and we can only hope that libraries will acquire a copy of the second edition as well. It is hard to write a book like *Visual geometry and topology*, and the author should be applauded for his efforts. When the new revised version appears, it should be a beautiful addition to any mathematical collection and something that can be recommended without reservation to mathematicians and students at all levels.

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