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Geometry of sets and measures in Euclidean spaces, by Pertti Mattila, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, 1995, x+343 pp., $49.95, ISBN 0-521-46576-1

The subject. Mathematical theories tend to deviate far from their origin. If you want to learn about measure theory and you look into a contemporary course, or into the AMS subject classification, you will hardly recognize that this field started with the determination of length, area and volume of sets in ordinary space. Yet there is a subfield called *geometric measure theory* which concentrates on the measurement of Borel sets in $\mathbb{R}^n$ – sometimes even of compact sets in the plane – and nevertheless embraces some of the deep problems and theorems of modern analysis, a few of them discussed in the present book.

For a long time, discoveries concerning ordinary space were considered rather as curiosities, and geometric measure theory played the rôle of Cinderella beside her big sisters, abstract measure and integration theory, who laid foundations for analysis and probability. Later, the measure-theoretic structure of sets in $\mathbb{R}^n$ turned out to be fundamental for analysis, too – both for the treatment of boundary value problems with singularities [6] and for the study of "exceptional sets" in various contexts [2]. Still, these were rather delicate and exclusive topics.

Meanwhile, complicated plane sets are commonly known as fractals due to the efforts of Mandelbrot [10] and to the expanding facilities of computer graphics, and it is generally accepted that such sets appear "everywhere" [1] in mathematics as well as in reality. Since paths of fractional Brownian motion and related fractal functions have come into fashion as models of stock market development and other everyday phenomena, there is a growing demand for results and exposition of geometric measure theory.

Pertti Mattila, one of the leading experts in the field and an excellent teacher, does not much care for fashion. He wants to get to the core of the subject, to the main ideas, in a way that readers can follow. Ten years ago he published a first version of his lecture notes [11] and since then has improved and extended the presentation. To see how the present text bridges the gap between the encyclopaedic monograph of Federer [6] and the more expository work of Falconer [4], [5], let us briefly and roughly sketch some basic concepts and history of the field.

Hausdorff measures. If a starting point of geometric measure theory can be named, it is the ingenious use of Caratheodory's measure construction by Hausdorff in 1919. Let the function f assign to each Borel set U a nonnegative number $f(U)$ as the size of U. For $\epsilon > 0$ let $m_\epsilon(A)$ denote the infimum of $\sum f(U_i)$ where the U_i cover A and have diameter $|U_i| < \epsilon$. Imagine A as a curve on checkered paper, and the U_i as the squares. We measure A by adding the sizes of the covering squares, and we increase our accuracy by passing to smaller squares. In fact $\mu(A) = \lim_{\epsilon\to 0} \mu_\epsilon(A)$ defines a measure on all Borel sets A.

The length measure for curves is obtained when $f(U)$ denotes the diameter $|U|$. A measure of area of A comes from $f(U) = |U|^2$, and, more generally, the

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t-dimensional Hausdorff measure $\mathcal{H}^t$ for arbitrary $t > 0$ comes from $f(U) = |U|^t$. For a given set A at most one value $\mathcal{H}^t(A)$ can be positive and finite – for larger t it will be zero and for smaller t infinite. This observation defines the Hausdorff dimension of A. Moreover, a length measure is also defined when $f(U)$ is the average length of projections of U onto lines through the origin. More generally, taking the average Lebesgue m-dimensional volume of projections onto m-dimensional linear subspaces, we define m-dimensional integralgeometric measure $\mathcal{I}^m$ for each positive integer m.

Regular and irregular sets. In his fundamental papers on plane sets written in the thirties, Besicovitch distinguished two classes of sets. The *regular or rectifiable* sets form a generalization of smooth curves, in a similar way as real functions of bounded variation generalize the smooth functions. On these sets $\mathcal{H}^1$ and $\mathcal{I}^1$ coincide, up to a normalizing factor. The *irregular* sets B with finite $\mathcal{H}^1(A)$ are those which do not contain a piece of a smooth curve, or, equivalently, those with $\mathcal{I}^1(A) = 0$. Today they are often referred to as *fractals,* and all A with $0 < \dim A < 2$ and with $\mathcal{H}^1(A)$ either zero or non-σ-finite are also assigned to this class. Each set of finite $\mathcal{H}^1$-measure divides into a regular and irregular part.

Two areas. It took twenty more years until Federer and Marstrand generalized Besicovitch's results to sets with finite $\mathcal{H}^m$-measure in $\mathbb{R}^n$. During the same time, analysis of smooth maps between manifolds was generalized to Lipschitz maps between rectifiable sets. Among others, differential forms and currents were defined, theorems of Gauss and Green derived and isoperimetric inequalities proved. In 1960 Federer and Fleming studied Plateau's problem on the existence and regularity of m-dimensional area-minimizing surfaces with given $(m-1)$-dimensional boundary in this generalized context. The field divided into two areas: structure theory of regular and irregular sets and analysis on rectifiable sets.

Federer's Book. *Geometric measure theory* [6], the first monograph of the field, came out in 1969. It is a monumental treatise with a wealth of material on both parts of the field as well as on related questions. Geometric properties of measures are studied as a tool to define a homological integration theory, culminating finally in a very general measure-theoretic calculus of variations. Just recently, in December 1996, Springer-Verlag reprinted the first edition in the new series Classics in Mathematics. With 690 pages for $35, this seems almost a gift to the mathematical community, compared with today's book prices.

Although still a valuable reference, [6] is not a textbook, however, due to its extremely condensed style. Various attempts were made to establish a more comprehensible presentation. Federer's 1978 survey [7] is a clear but still advanced introduction to his book, with hints to new developments. Morgan's beautifully illustrated *Beginner's guide* [13], written for a wider audience, and the comprehensive text by Simon [17] also lead the way to variational problems. Related material on harmonic maps and on applications in computer vision was recently discussed in the *Bulletin* [8], [14].

Falconer and Mattila. The text under review concentrates on the geometric part of the field, indicated by the subtitle "Fractals and Rectifiability" – although "Rectifiability and Fractals" would be more appropriate, as we shall see below. Compared to [6], this book reads like a novel, but it is definitely a graduate text. For an introduction, the successful books by Falconer [4], [5] are recommended.

[4] presents Besicovitch's results on subsets of the plane, quoting higher-dimensional generalizations without their complicated proofs. In the last chapter, mathematical constructions of fractals are briefly discussed. Falconer's *Fractal geometry* [5] from 1990 gives a rigorous treatment of Mandelbrot's concept of fractals accessible with undergraduate mathematical background. Results of geometric measure theory are discussed here in the context of calculating fractal dimension, and classes of fractals like self-similar sets, strange attractors, Julia sets, and fractional Brownian motion are treated in separate chapters. Other mathematical textbooks on fractals like Barnsley [1] and Edgar [3] concentrate on detailed studies of such particular classes of sets obtained by recursive constructions.

Falconer provides a clear understanding of the basic concepts; Mattila goes further and leads the reader straight to the edge of current research.

Three levels of presentation. *Essential results* are treated carefully, in complete detail, in n dimensions, with lucid exposition of the most important proof techniques, for instance capacities, Fourier transforms, tangent measures. This material was selected with a fine feeling for clarity, relevance and beauty. No attempt for maximum generality of statements or maximum number of theorems is made. For instance, Hausdorff measures can easily be treated in complete separable metric spaces, as in the book of Rogers [16], but here we stay in $\mathbb{R}^n$. (There is one exception in chapter 8, where an elegant new approach by Howroyd to Frostman's lemma and to the existence of subsets of finite Hausdorff measure in compact metric spaces is presented.)

Convincing inner-mathematical applications illustrate the use of the introduced concepts and methods and their connection with other subjects, as porosity, dimension of Brownian paths, removable sets and singular integrals.

Finally, there are *a huge number of remarks and detailed references* concerning past and recent work on related topics. The bibliography contains almost 500 titles, half of them from the nineties. Among others, all specific classes of fractal constructions as well as harmonic measures and Salem sets are treated in this group of "further results". Take for example the theory of multifractals, a subject not yet in its final stage but currently of great importance for physical applications. Mattila introduces on one page the basic notion and briefly quotes about 20 references. Probably this topic deserves more attention, but the author wanted to avoid a too specialized or speculative treatment in the style of [10], [1]. Nevertheless, people like Lars Olsen who presently work on multifractal theory make use of Mattila's proof techniques.

A sample of results. The state of the art is that there are "rough" statements, on Hausdorff dimension, for arbitrary sets, and "more precise" statements for sets A with $0 < \mathcal{H}^t(A) < \infty$. We fix such a set A to give a flavor of the contents of the book. The basic concept for studying the structure of A is the *t-dimensional density* of A at a point x,

$$D^t(A,x) = \lim_{r\to 0} \frac{\mathcal{H}^t(A \cap B(x,r))}{(2r)^t}$$

where $B(x,r)$ denotes the ball of radius r around x. This limit need not always exist, in which case we consider the upper and lower limits $\overline{D^t}(A,x)$ and $\underline{D}^t(A,x)$, respectively. The density of A at $\mathcal{H}^t$-almost all points x outside A exists and is zero. Call a point x inside A regular if the density of A at x exists and equals one.

A is said to be regular if almost all points x in A are regular. A is called irregular if almost no point x in A is regular. For $t = 1$ these are Besicovitch's original definitions.

Marstrand proved that for non-integral t, all sets A are irregular. In popular terminology: "fractional dimension implies fractal structure". For integral t, regular and irregular sets are distinguished by the size of their projections, as mentioned above. For all t, the upper density $\overline{D^t}(A, x)$ lies between 2^{-t} and 1 for almost all x in A. That means, for almost all x we can find arbitrary small r such that the t-dimensional Hausdorff measure of A inside $B(x, r)$ is larger than r^t.

Here are two more delicate results of Mattila for $n - 1 < t < n$. Let us fix a unit vector $y \in \mathbb{R}^n$ and an arbitrary small angle $\alpha > 0$. We can define a cone $C(x, r)$ with vertex x, radius r, opening angle α and direction of axis y as a small spiky subset of $B(x, r)$. Then for almost all x in A there are arbitrary small r such that $\mathcal{H}^t(A \cap C(x, r)) \geq cr^t$. The constant c depends only on α, not even on y. Thus, roughly speaking, the fractal measure $\mathcal{H}^t$ restricted to A is scattered completely around each point x in A, in every neighbourhood of x.

On the other hand, a given unit vector y defines a hyperplane through x which divides $B(x, r)$ into a positive and a negative hemisphere. Mattila shows that for each $\epsilon > 0$ and almost all x in A, there exist some y such that $\mathcal{H}^t(A \cap B^+(x, r)) \leq \epsilon r^t$ for arbitrary small r. Thus, in some weak sense, we have locally supporting hyperplanes to A. In conjunction with the previous result this gives a feeling for the tremendous fluctuations of fractal measures near almost every point.

Rectifiability is the main theme of the book. A set A in $\mathbb{R}^n$ is m-rectifiable if it is the union of a $\mathcal{H}^t$-zero set and countably many sets of the form $f_i(C_i)$ where the C_i are sets in $\mathbb{R}^m$ and the $f_i : C_i \to \mathbb{R}^n$ are Lipschitz maps. It is obvious that this generalizes the definition of smooth submanifolds, but it is not obvious that each regular set of dimension m is in fact m-rectifiable. 1986 D. Preiss [15] proved a much stronger conjecture which had been open for a long time: *Whenever the m-dimensional density of A exists as a positive finite number, for almost all x in A, then A is m-rectifiable.*

The author gives an accessible account of this profound result. Only for the two-dimensional case is an almost complete proof given. A most important new tool, however, the concept of *tangent measures,* is worked out well and applied to several other topics. While a density describes the limiting amount of mass of A in a small neighbourhood of x, a tangent measure describes a limit structure of A (or of a measure μ) in a small neighbourhood of x.

There are many properties equivalent to rectifiability, some of them quite surprising. In the last chapter, the author shows that rectifiable sets can be characterized by the existence of principal values of certain singular integrals. Chapter 19 treats Ahlfors' analytic capacity in the complex plane. A set $A \subset \mathbb{C}$ is removable if for every open $U \supset A$, every bounded analytic function $f : U \setminus A \to \mathbb{C}$ has an analytic extension to U. It is known that A is removable when $\mathcal{H}^1(A) = 0$ and not removable for $\dim A > 1$. A conjecture states that for finite $\mathcal{H}^1(A)$, removable and irregular sets coincide. Meanwhile, this has been proved under quite general conditions by the author, Melnikov and Verdera in a remarkable paper [12].

Challenge. Today, geometric measure theory is faced with a two-fold challenge. On one hand, sets beyond those of finite $\mathcal{H}^t$-measure have to be investigated. It is unlikely that much can be proved for the structure of very inhomogeneous sets, but

the present theory is too narrow. Trajectories and zero sets of Brownian motion, harmonic measure and other interesting examples cannot be measured by $\mathcal{H}^t$; they need Hausdorff measures with logarithmic corrections [18]. On the other hand, it should be possible to develop analysis not only on rectifiable, but also on irregular sets. Quite a number of recent papers address this issue (see for instance [9]), but so far the basic spaces are very special recursively generated fractals.

The present textbook paves the way for further research in these directions. It will be a reference for many years to come.

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