BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, Number 2, April 1998, Pages 179–188
S 0273-0979(98)00748-4

Three-dimensional geometry and topology, by William Thurston, Princeton Univ. Press, Princeton, NJ, 1997, v+311 pp., $39.50, ISBN 0-691-08304-5

No mathematical subject lies closer to intuition than the geometry of two and three dimensions. Few people today would defend Kant's idea of the a priori inevitability of Euclidean geometry. But the primacy of space in our perception of the world remains an unquestionable fact of psychology.

It often happens that mathematical subjects with strong roots in intuition are actually considerably more difficult than those wherein the structures are more artificial. We are at liberty to choose which of the more exotic infinite dimensional vector spaces we want to study. We have no choice about finite groups or whole numbers or low-dimensional manifolds. Even within topology, the manifolds forced upon us, those of low dimensions, have turned out, in the cases of dimensions 3 and 4, to be more intractable than those of higher dimensions that are more nearly of our own creation.

Even so, and rather surprisingly, there seems to be some hope of understanding 3-manifolds by a geometric method, by what amounts to geometric intuition, albeit pushed very far. The book under review is the first volume of an introduction to Thurston's geometric program for understanding 3-manifolds, analogous to the uniformization of 2-manifolds but by nature far more complicated and profound. Even in its present partial form, this program contains mathematics of great power and significance. If and when it is completed, it will be one of the great monuments of mathematical thought.

The book is a reworking of the first part (the first four chapters, up to the chapter on orbifolds) of the well-known lecture notes of Thurston on the subject that have been widely circulated in various versions for a long time. However, the book is something quite different from any version of the notes. The material has been revised, expanded, and reworked by Thurston with the editorial assistance of Silvio Levy to the point of transformation. No one should suppose that the lecture notes, intriguing though they were and are, are a substitute for the present book and the volume(s) yet to come. On the other hand, the present volume at least is considerably more elementary than the more advanced parts of the notes (e.g., hyperbolic Dehn surgery in generality is omitted).

To put Thurston's general program and the present book in perspective, one needs to recall the situation for surfaces, e.g., 2-manifolds. Of course surfaces have been the object of geometric study for thousands of years, in more or less specific cases. But the key to understanding the geometric theory of all surfaces at once arose from complex analysis and came along rather late in the day: The final result depends on the Koebe Uniformization Theorem, which was obtained only in 1907. With the benefit of hindsight to make things as efficient as possible, one proceeds as follows:

Suppose M is an orientable (C^1 paracompact) 2-manifold. Then M admits an analytic Riemannian metric, analytic, that is, relative to *some* compatible real analytic structure: This follows by choosing a real analytic approximation of the

1991 *Mathematics Subject Classification.* Primary 57M50; Secondary 53C30.

image of a C^1 embedding, a possibility pointed out by H. Whitney[1] for manifolds in general [Wh]. Then it can be seen from partial differential equation theory that the surface admits "local isothermal parameters", i.e., local coordinates (x_1, x_2) such that the metric has the form $(\lambda^2(x_1, x_2))(dx_1^2 + dx_2^2)$ for some (real analytic) function $\lambda(x_1, x_2)$. Actually, much less regularity of the metric than real analyticity is required here, but the real analytic case is much easier and was done much earlier.

Choosing a covering by coherently oriented "isothermal" local coordinate patches produces on M a Riemann surface structure. That is, the transition from one such coordinate system (x_1, x_2) to another (x_1', x_2') is holomorphic when thought of as a complex mapping $x_1 + ix_2$ to the corresponding $x_1' + ix_2'$.

Now one can "lift" the Riemann surface structure on M to the simply connected universal cover $\hat{M}$ of M. Historically, the whole idea that this previous sentence summarizes took a long time to develop on anything resembling a firm basis, and the precise theory of covering spaces ranks as one of the great accomplishments of mathematics of a century or so ago, however much we all take it for granted today.

In this setup, M can be thought of as the quotient of $\hat{M}$ by the group Γ of covering transformations. A priori, the group Γ acts holomorphically, i.e. conformally, and no more. But at this point, the facts of concrete mathematics supplement the general picture to introduce true, metric geometry, not just conformal mappings, into the situation:

According to Koebe's celebrated Uniformization Theorem already mentioned, the simply-connected Riemann surface $\hat{M}$ is biholomorphic to either $\mathbb{C}P^1$, the Riemann sphere; or $\mathbb{C}$, the complex plane; or D, the unit disc in the complex plane. Now Γ acts on $\hat{M}$ via fixed-point-free biholomorphic maps. And it is just a fortunate specific fact that the fixed-point-free biholomorphic maps are isometries of some metric in all three cases.

For $\mathbb{C}P^1$ the situation is particularly simple: There are no non-identity fixed-point-free biholomorphic maps, so Γ must consist of the identity alone. Reason: The biholomorphic maps are the linear fractional transformations, and one sees by calculation that they all have fixed points. Or one can use the fact from topology that any orientation-preserving homeomorphism of S^2, being of degree 1, has a fixed point; this is a bit like shelling a walnut with a sledge hammer, however.

For $\mathbb{C}$, the biholomorphic maps must be linear $z \to az + b$ and then, to be fixed-point-free, must be translations, $z \to z + b$, preserving the standard metric of $\mathbb{R}^2$.

For D, the situation is a little more subtle: The biholomorphic maps are the Möbius transformations $z \to w(z-a)/(1-\bar{a}z),\ |w| = 1,\ a \in D$. And it turns out that all these (fixed-point-free or not) act as isometries of a metric, namely, the "Poincaré metric" $4(dx^2 + dy^2)/(1 - x^2 - y^2)^2$ in Euclidean (x, y) coordinates on D. With hindsight, it is more or less obvious that such an invariant metric exists,

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[1] As it happens there is a significant misapprehension on this point in the book, p. 113, where it is asserted that Whitney [Wh] establishes the uniqueness of a compatible real analytic structure, as well as existence. Whitney's paper does prove existence, but uniqueness requires the far more profound results of Grauert [Gra] and Morrey [Mor] on the existence of embeddings in some Euclidean space that are real analytic for a *given, fixed* real analytic structure. This is quite different from and far more difficult than Whitney's observation that a C^1 submanifold of $\mathbb{R}^n$ can be approximated by a real analytic submanifold. This distinction plays no role anywhere else in the book, but it is so widely misunderstood altogether that it seemed worth pointing out. See [GrW] for a further look at the history.

simply because the isotropy of the origin, and hence of any point (D is biholomorphically homogeneous) is compact. In fact, one finds the invariant metric simply by noting that it must be rotationally invariant at the origin since rotations act biholomorphically, so at the origin it is a multiple of the Euclidean metric, and what the invariant metric is elsewhere is determined by the fact that D is homogeneous under the Möbius transformations. This resulting metric homogeneity also implies that the invariant metric has constant Gauss curvature - negative, as it must happen. (If it were 0, then D would be isometric in the conformal invariant metric to $\mathbb{C}$ and hence biholomorphic to $\mathbb{C}$, contradicting Liouville's Theorem; if it were positive, then D, being homogeneous and hence complete, would have to be compact.)

Returning now to the manifold M itself, one has a gratifying conclusion: M admits a complete metric of constant Gauss curvature. This follows from the fact that Γ acts on $\hat{M}$ as isometries so that the constant curvature metric of $\hat{M}$ "pushes down" to M. The "push-down" has constant curvature, being locally isometric to $\hat{M}$, and is complete because a covering space quotient of a complete metric is always complete.

If M is compact, which sign of curvature a constant curvature metric on M can have is uniquely determined by the topology of M. For instance, the Gauss-Bonnet formula $\int K dA = 2\pi\chi(M)$ shows that the sign of the curvature (or zeroness) is the same as for the Euler characteristic.

When M is noncompact, then ambiguity can arise. The punctured plane $\mathbb{R}^2\backslash\{(0,0)\}$ can be given a complete metric of zero curvature and also a complete metric of constant negative curvature. These correspond, respectively, to $\mathbb{C}$ covering $\mathbb{C} - \{0\}$ by $z \to e^z$ and to D covering, say, $\{z \in \mathbb{C} : 1 < |z| < 2\}$. Geometrically, one can put metrics on $\mathbb{R} \times S^1$ of the form $dr^2 + d\theta^2$ for zero curvature and $dr^2 + (\cosh^2 r)(d\theta)^2$ for constant -1 curvature.

The punctured plane or, equivalently, $S^1 \times \mathbb{R}$ is, however, the only ambiguous case. It is easy to see that $\mathbb{C}$ covers only $S^1 \times S^1$ and $S^1 \times \mathbb{R}$ if the covering transformations are required to be translations. Thus $S^1 \times \mathbb{R}$ is the only noncompact manifold arising from a holomorphic quotient of $\mathbb{C}$. All other noncompact topological possibilities arise only from quotients of D. Of course this viewpoint also recovers the compact result: S^2 from $\mathbb{C}P^1$ (positive curvature), $S^1 \times S^1$ from $\mathbb{C}$ (zero curvature), all others from D - except that one needs to make an argument that $S^1 \times S^1$ cannot arise from a holomorphic quotient of D, e.g. as a special case of Preissman's theorem on abelian subgroups of the fundamental group of a manifold of constant negative curvature. (This related to later developments relating growth of fundamental group to volume growth in the universal cover [Sw].)

The whole picture has a perfect elegance worthy of classical Greek geometry: Every surface, no matter how complicated, admits a geometry with the greatest possible local symmetry. And except in one case - the cylinder - where two geometries coexist, which local geometry occurs is uniquely determined by the topological type of the surface.

It is important to observe, however, that the constant curvature metric is not itself uniquely determined, in general - only the *sign* of the curvature is nailed down by the topology. Of course, a trivial variation is possible, since a constant positive multiple λg of a metric g of constant curvature K has again constant curvature $\lambda^{-1}K$. But nontrivial variations of the metric are possible in general. For S^2

and for 0 curvature on $S^1 \times \mathbb{R}$, the constant curvature metrics are unique up to constant multiples and isometry. But in other cases, nontrivial variations exist in positive-dimensional families. These variations of the complex structure on, say, a compact Riemann surface of genus $g \geq 2$ give rise to a family of nonisometric metrics of constant curvature -1. It was determined long ago by Riemann that the dimension of the family here is $6g-6$ (real dimensions). For $g = 1$ (the torus), the real dimension is 2. This developed historically into a subject of great interest unto itself, of course.

For compact surfaces, it is possible to bypass complex analysis and deal with the construction of constant curvature metrics more directly. First one proves the topological classification result that every surface is obtained by identification of edges of a suitable polygon. Then one shows that this identification process can be carried out metrically as it were, i.e., in either flat $\mathbb{R}^2$ or the hyperbolic plane or S^2, in such a way as to produce a smooth constant curvature metric on the surface. This is in fact the approach used in the present book, which does not deal with uniformization in the complex variables sense.

In view of the enormous success of this program of understanding 2-manifolds by "uniformization" by constant curvature, it is rather surprising how long it was before substantial progress was made in extending the program to higher dimensions. This is not to say that either topology of manifolds nor Riemannian geometry were languishing subjects meanwhile. The study of manifolds of dimension five and greater by differential-topological methods was of course one of the triumphs of mathematics in the fifties and sixties. And, with its re-energizing by H. Hopf's ideas of curvature and topology in the twenties and later, Riemannian geometry entered a "boom" which still continues. But for a long time, geometry and topology went surprisingly separate ways.

To see the logic of this separation, one needs to look at what was happening in Riemannian geometry. The 2-dimensional situation certainly suggested guiding principles for Riemannian geometry. For instance, the uniqueness of the 2-sphere among (orientable) 2-manifolds of positive curvature gave rise to a general view that there would be few manifolds of positive curvature in higher dimensions, in some sense of the word *few*. This was given formal substance in the theorem of A. Weinstein [We] that there are only finitely many homotopy types among manifolds of a fixed even dimension with sectional curvature lying in an interval $[\delta, 1]$, $\delta > 0$. This theorem led to a sub-industry of "finiteness theorems" accompanying the inevitable sub-industry of finding geometric conditions under which a manifold must be a sphere. (The most famous of these sphere theorems was the "1/4-pinched" Berger-Klingenberg theorem that compactness, sectional curvature $\in \left(\frac{1}{4}, 1\right)$ and simple connectivity imply homeomorphism to a sphere.) In the complex case, a uniqueness held exactly analogous to the unique position of $\mathbb{C}P^1$: A compact Kähler manifold of positive sectional curvature is biholomorphic to $\mathbb{C}P^n$. This result, long conjectured and the subject of many partial results, was finally proved as stated by Mori [Moi] via algebraic-geometric methods and Siu-Yau [SY] via complex geometry.

In the negative curvature case, progress was similarly rapid. But unexpected new features arose. First, there was the rigidity result of Mostow that a compact manifold of dimension ≥ 3 could have at most one metric structure with sectional curvature identically -1. Second, following the pioneering work of Preissmann and

Bishop-O'Neill, a great deal of information was derived concerning the possible fundamental groups of manifolds of negative sectional curvature. (It had been known for a long time that such a manifold was topologically a quotient of a Euclidean space by some covering transformation group so that the fundamental group and its action on the universal cover embodied all the topology of the manifold.) In the locally determined zero sectional curvature case, the situation is again that the universal cover is $\mathbb{R}^n$, but now it is also metrically $\mathbb{R}^n$ with its standard metric, and the possible fundamental groups are highly restricted. These were classified by Bieberbach.

The very success of these developments discouraged thoughts of uniformization. Rather few manifolds of higher dimension than two seemed to admit metrics with sectional curvature of one sign, much less of constant sectional curvature. As examples, one needs only to find n-manifolds with universal cover different from $\mathbb{R}^n$ but with first Betti number greater than n, since nonpositive curvature implies universal cover $\mathbb{R}^n$ while nonnegative curvature had been shown by S. Bochner to imply first Betti number $\leq n$. For instance, it suffices to look at the connected sum of: (1) a product of S^2 and $(n-2)$ S^1 factors and (2) a product of a high-genus Riemann surface with $(n-2)$ S^1 factors. (The presence of homotopically nontrivial S^2 precludes universal cover $\mathbb{R}^n$, while the 1-homology generators from the Riemann surface persist and give first Betti number $> n$.)

A further discouragement was of a practical nature: It is relatively easy, using hyperbolic geometry, to construct (compact) quotients of the Poincaré disc of high topological complexity: One is dealing only with lengths, angles, and reflections in geodesics. But in dimension 3 and higher, these direct constructions become much harder. In fact, even today, few constructions are known in dimension ≥ 4 other than by methods of arithmetic groups.

Of course, a sufficiently perceptive, or optimistic, person might have taken heart at least from the fact that all but one of the "bad" examples of the previous paragraphs are connected sums of rather "good" examples, simple products of objects of standard geometries. But in fact, the whole matter was quiescent until Thurston's discovery that a great many 3-manifolds admitted hyperbolic structures, i.e., complete metrics of constant negative curvature. In the aftermath of this discovery and related developments, Thurston proposed a substitute for literal uniformization. Namely, he conjectured that in a certain sense every compact 3-manifold can be obtained as a finite union of pieces admitting "geometric structures". The geometric structures are what correspond in dimension three to constant curvature in dimension two, in a sense to be explored further momentarily. While it will be easy to describe what it means for a compact manifold to admit one of these geometric structures, the precise sense in which a general 3-manifold can be (conjecturally) decomposed into geometrically structured pieces is somewhat subtle and in complete generality is outside the scope of a summary review. Peter Scott's article [Sc] gives a clear and detailed exposition for the reader wanting to look at details here. (One easily stated specific form is: If M is a closed, oriented prime 3-manifold, then there is a finite set of disjoint embedded 2-tori $\{T_j\}$ such that each (open) component of the complement in M of $\cup\, T_j$ admits a geometric structure in the sense of admitting a complete metric, the (necessarily complete) universal metric cover of which is one of the eight "model geometries" to be defined later in this review.)

"Geometric structures" are in effect the simply connected Riemannian homogeneous spaces with the additional property that they admit compact quotients. The classification of Riemannian homogeneous spaces of dimension three goes back (at least) to E. Cartan's 1928 book [Car] as an application of his general theory of moving frames and differential forms. (Here in this book it is worked out by hand, as it were, and no historical attributions of the result are given.) There are eight of these geometric structures (in dimension 3). First, there are the three simply connected "space forms", that is, the simply connected complete Riemannian manifolds of constant $+1, 0$ and -1 curvatures, namely, S^3, $\mathbb{R}^3$ (called in the book E^3), and H^3 (the hyperbolic 3-space). (Here metrics that differ by constant multiples are regarded as essentially equivalent, so $+1, 0, -1$ are the only constant sectional curvature possibilities.) Next are the metric products of $\mathbb{R}$ with the 2-dimensional simply connected space forms, namely, $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$. (Of course, $\mathbb{R}^2 \times \mathbb{R}$ is $\mathbb{R}^3$, which has already been listed.)

The remaining three geometries are less directly geometric in a sense, and more Lie group theoretic. They are the universal cover of the unit tangent bundle of H^2 denoted $\widetilde{SL}(2, \mathbb{R})$, together with two additional structures that are dubbed nilgeometry and solvegeometry: These are, respectively, the geometry of the Heisenberg group (the group of upper triangular matrices with ones on the diagonal) and the solvable Lie group of dimension 3 consisting of the transformations given by $(x, y, t) \to (e^{t_0}x + x_0,\ e^{-t_0}y + y_0, t + t_0)$ parametrized by $(x_0, y_0, t_0) \in \mathbb{R}^3$.

A compact 3-manifold can be uniformized by at most one of these eight geometries. That is, given a compact 3-manifold M, there is at most one of the eight model manifolds with the property that the manifold M is a quotient of the model by a discrete (and cocompact) subgroup of the isometry group of the model. Moreover, which model can occur can be determined explicitly from the fundamental group $\pi_1(M)$ as a group. For example, if $\pi_1(M)$ is finite, then the S^3-with-constant-positive-curvature model is the only possibility. If $\pi_1(M)$ is almost abelian (has a finite index abelian subgroup), then the possible models are $S^2 \times \mathbb{R}$ and $\mathbb{R}^3$, the alternative depending on whether $\pi_1(M)$ is almost cyclic or not. If $\pi_1(M)$ is not almost abelian but is almost nilpotent, then the model must be nilgeometry. If not almost nilpotent but almost solvable, then solvegeometry. If not almost solvable and with normal infinite cyclic subgroup, then hyperbolic space. If not almost solvable but with a normal infinite cyclic subgroup, then the quotient group by the normal infinite cyclic subgroup is a cocompact discrete group of hyperbolic isometries. If this latter group has a subgroup of finite index that splits, then the model is $H^2 \times \mathbb{R}$. If not, the model is $\widetilde{SL}(2, \mathbb{R})$.

This classification is by no means obvious. Indeed, it incorporates the 3-dimensional case of, for instance, Preissmann's well-known theorem that the fundamental group of a compact manifold of negative sectional curvature contains no abelian, noncyclic subgroup.

This classification is the concluding result of the book, and it, together with the classification of the eight geometries, is the book's main goal. Rather surprisingly, these results, which lie quite far into the theory of Riemannian homogeneous spaces, are reached in detail even though the book starts almost from the very beginning of geometry and even of manifold theory. The first two chapters of the book deal with surfaces and their uniformization and with hyperbolic geometry, in two and three dimensions especially. The third chapter treats geometric structures on manifolds in general and presents the classification of the eight three-dimension geometries. Along the way, there is a good deal of manifold topology discussed along the lines of triangulation ideas, constructions by gluing on so on. (Many of those results alluded to are not proved but discussed informally as background for the main line of development.) The fourth chapter discusses discrete groups and culminates with the classification associating fundamental group and model geometry already described. Roughly speaking, the contents of the book are similar to that of the expository survey of Thurston's program by Scott [Sc]. But the treatment in the book is much more ab initio. In a sense, the book is an unusual kind of textbook with (formally) rather elementary prequisites considering how far it gets. Next to no pre-existing knowledge of anything at all is presumed on the part of readers, although a very great deal of either determination and/or sophistication is evidently required.

The book is in fact written in an extraordinary fashion. The opening sentence of the preface is "The style of exposition in this book is somewhat experimental." And after reading the book, one is inclined to regard that as an understatement. I would suppose that almost everyone has come to believe that the definition-theorem-proof style epitomized by Bourbaki is not the best way to present mathematics and that more motivation and more examples presented in detail give the reader far better understanding in the end. This book represents, however, a really extreme form of this growing belief. For one thing, many of the absolutely fundamental concepts are given little or no formal definition or development. This is in effect a book on (part of) Riemannian geometry that defines the concept of Riemannian metric very briefly in an appendix and defines sectional curvature only by examples!

The natural question to ask is, does this method work? For someone who already knows basic manifold topology and Riemannian geometry, I have no doubts that the book will be instructive and inspiring. And it will bring home strongly the fact that there is a great deal more to know about these subjects than one finds out in the "freeway tour" that many texts offer. How many students come out of a usual geometry course knowing how to construct any compact hyperbolic 3-manifold whatever? Even though the Seifert-Weber example presented here was discovered in the thirties, such concrete examples are often left out of sight. The resulting absence of concrete and complete understanding can be costly and lead to peculiar situations, for example, where high flying differential equations methods have recovered results of Riemannian geometry without the rediscoverers being aware the results were already known.

On the other hand, to approach this book from, say, just an undergraduate course on topology (which would suffice in principle) would require herculean effort and dedication. And it would also lead to a somewhat narrow view of the subject of Riemannian geometry, especially. General manifold topology gets a broader view; many of the major results are mentioned at least tangentially. The book is surely more accessible by far than the lecture notes, some versions of which required substantial effort even for people with lots of background. But the present book still seems very demanding, in a somewhat different way, for beginning readers.

To take a random sample, in the presentation of the Seifert-Weber hyperbolic dodecahendral space already alluded to and the related spherical (Poincaré) dodecahedral space, it is taken as apparent without comment that if the dihedral angles are such that the metrics on the 3-dimensional interiors fit together along edges, this will also yield a nonsingular metric at the vertices. This is true: Isolated metric singularities of the type one wishes to avoid are in fact not possible in constant curvature in dimension three or greater. But the proof of this requires more insight than a beginner seems likely to have reached by page 37. (The detailed treatment in, e.g., [Ra] is quite involved, for instance, although one can in fact deal with the matter more easily by using continuation of local isometries and the fact that $\mathbb{R}^3$-{point} is simply connected.)

A second pecularity is that, while the book is in a sense recapitulating the historical and hence the psychological development of the subject, historical references are rather sporadic. This is not terribly important: A text need not be a history book. But the level of historical reference fluctuates oddly. On the one hand, one is left in doubt about such basic and central matters as the original source of the classification of the eight geometries. (The treatment in [Sc] is also devoid of reference, e.g., to Cartan.) On the other hand, certain peripheral matters are given detailed references. There are odd omissions in details: Double suspension is referred to without reference to R. Edwards in the index or bibliography, although he is referred to in the text. (On a personal note, J. Moser's result on the action of diffeomorphisms on volume forms on compact manifolds is discussed without reference to the corresponding result in [GrS] of K. Shiohama and the present reviewer for noncompact manifolds.) And, as noted in the earlier footnote, at least one historical description is incorrect in a mathematically substantial way. But the most disconcerting aspect of all this is the absence of references to the large results from geometry compared to the small topological points discussed in detail: E. Cartan and Preissmann are missing, while many small contributions by topologists are referred to carefully. An author is certainly allowed to emphasize what is considered important, but the book is in good part about geometry, even so.

Returning to the mathematics, it is worth pointing out that the actual proof of Thurston's geometrization conjecture may well come from some direction entirely different from the purely geometric and topological one. There are partial differential equations methods proposed (by R. Hamilton [Ha 2] via Ricci flow, by M. Anderson [An] via the solution of the Yamabe problem) that may triumph where more intuitively geometric methods have not.

In a way, this is not surprising. The uniformization of compact Riemann surfaces (of genus at least two) can be thought of as a special case of the Calabi conjecture (in the negative sign case), and as such can be approached and proved by partial differential equations methods. (In fact, the Riemann surface case is much easier in this viewpoint than the methods of T. Aubin and S. T. Yau used for the higher dimensional situation.) Moreover, Yamabe's original ideas and proposed proof of his conjecture for constant scalar curvature were, it seems, motivated by the Poincaré conjecture. The well-known results of Hamilton [Ha] on compact 3-manifolds of positive Ricci curvature are related even more directly to geometrization. (Hamilton proved that on such a manifold there is a metric of constant positive sectional curvature so that the manifold is a quotient of S^3 by a discrete, finite linear action. This simultaneously disposes of the Poincaré conjecture for the universal cover and the question of equivalence of finite fixed-point free group actions to linear ones in this particular case. Conversely, if the topological conjectures were known, Hamilton's result would be recovered without using his Ricci flow method.)

In any case, it remains to be seen whether partial differential equations methods can be used to produce any purely topological results in this situation. Certainly any significant topological results here by these methods would constitute a vindication of the utility of those geometric analysis methods which have proved so outstandingly useful in complex manifold theory without so far being comparably useful in manifold theory in general.

Beyond any small matters of detail about which one might have some reservations, the book is overall something of remarkable, even unique, value. It is hard to think of another contemporary mathematics book that gives readers so much insight into how its author actually thinks about its subject. All mathematicians have a private world of which their publicly presented proofs and theorems are but a formal representation. (Einstein once wrote of how his discoveries came to him in the form of strange visual images, meaningless to others, that had to be translated into the ordinary language of physics and mathematics before they could be presented publicly at all.) In this book, one has the feeling of being allowed a look at how the author, who is of course possessed of unique insight into the subject of 3-manifolds, actually thinks about them. Most books allow one to meet the author in a lecture hall. This book is, one feels, a visit to his home, mathematically speaking. So far, in this first volume, only relatively elementary matters have been treated, many of which were already quite accessible in the expository literature, though not with the same sense of being present at some of the creation. Even so, the result is an intriguing work that offers not only interesting mathematics but also by example new ideas on how to write mathematics. If later volumes can similarly convey to the reader the same level of insight and intuition about the more difficult and less well-known (or even previously unknown?) matters to come, the result will be a true classic of mathematical writing.

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