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Cohomological induction and unitary representations, by A. W. Knapp and D. A. Vogan, Jr., Princeton Univ. Press, Princeton, NJ, 1995, xvii +948 pp., ISBN 0-691-03756-6

Representation theory is very useful in understanding problems in real complex analysis, number theory and automorphic forms. A representation (π, V) of a group G is a group homomorphism

$$\pi : G \to \mathrm{Aut}(V)$$

where V is a complex vector space. For analysis applications, G is a locally compact topological group, V is a locally convex space, and the corresponding map $G \times V \to V$ is assumed to be continuous. The representation is called irreducible if V has no proper nontrivial G-invariant closed subspace. We will assume that the underlying space V is a Hilbert space. The representation is called unitary if in addition $\pi(g)$ is unitary for all $g \in G$.

The situation that arises quite often is the following. Let H be a closed subgroup of G. Then complex valued functions on G/H form a representation via $\pi(g)f(x) := f(g^{-1}x)$. If G/H admits a G-invariant measure, then $L^2(G/H)$ is a unitary representation. The problem is to decompose this representation under the action of G. For example if $H = SO(n-1)$ and $G = SO(n)$, then G/H can be identified with the unit sphere S^{n-1}. The decomposition of $L^2(S^{n-1})$ is related to spherical harmonics. There are many such relations between representation theory and special functions. A comprehensive reference is [KV].

The representation on $L^2(G/H)$ generalizes as follows. Let ρ be a representation of H. We can form the *induced representation* $(\iota_\rho, \mathrm{Ind}_H^G(\rho))$ where

$$\mathrm{Ind}_H^G(\rho) = \{f : G \to V_\rho : f(gh) = \rho(h)^{-1} f(g)\}, \quad \iota_\rho(g)f(x) = f(g^{-1}x).$$

One of the basic questions of representation theory is to classify the irreducible representations of a group G. For finite groups, the well known character theory implies that there are as many inequivalent irreducible representations as there are orbits under the adjoint action $\mathrm{Ad}(g) \cdot x := gxg^{-1}$. It is tempting to try to implement an explicit parametrization. It is not clear how to do this in such generality. One possible simplification could be provided by using induced representations. For each group we choose a special set of representations which we call cuspidal. Then for an arbitrary group, every representation should be obtained by induction from cuspidal representations of certain subgroups. The first problem with this is that induced representations are rarely irreducible. This might be remedied by trying to single out a canonical subquotient. But more serious is that one has to find the right kind of subgroup and definition of cuspidal.

These notions have their origin in the work of Harish-Chandra and Langlands. They proved very useful for groups which are rational points of connected reductive linear algebraic groups. A connected linear algebraic group G defined over some field k is called reductive if its unipotent radical (the maximal connected unipotent subgroup) is trivial. In this case we only use certain subgroups called *parabolic*

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subgroups. If $G = GL(n)$, a parabolic subgroup is one that is conjugate to a block upper triangular group. Parabolic subgroups have a decomposition as $P = MN$ where N is the unipotent radical and M is again a reductive group. This is called a Levi decomposition and M a Levi factor. Then any representation ρ of M can be inflated to a representation $\tilde{\rho}$ of P by making it trivial on N. The induced module $\mathrm{Ind}_P^G(\tilde{\rho})$ is called the *Harish-Chandra* induced module. A representation of G is called *cuspidal* if it does not occur as a composition factor in any module which is Harish-Chandra induced from a representation on a proper Levi component. For the finite groups of Lie type or Chevalley groups (e.g. $GL(n)$ over a finite field),

$$\mathrm{Hom}_G[V_\pi : \mathrm{Ind}_P^G(\tilde{\rho})] \cong \mathrm{Hom}_M[V_\pi^N : \rho], \quad V^N := \{v \in V : \pi(n)v = v \ \forall n \in N\}.$$

So V is cuspidal if and only if $V^N = \{0\}$ for any unipotent radical of a proper parabolic subgroup. The classification problem reduces to the (nontrivial) problems of describing the cuspidal representations and of decomposing Harish-Chandra induced modules from cuspidal representations.

When G is a (real) Lie group, it is important that the representation be differentiable so that we can get a representation of the Lie algebra $\mathfrak{g}$. Such representations are called *smooth*. A basic theorem of Lie theory states that finite dimensional representations are always smooth. For compact Lie groups irreducible representations are finite dimensional, therefore smooth. But compact groups do not have any interesting parabolic subgroups, just the identity group and the group itself. Still, irreducible representations can be classified in the spirit outlined earlier. The relevant result is called the theorem of the highest weight. As we shall see below, it can be thought of as a variant of induction. Assume that G is connected, and let $T \subset G$ be a maximal torus. Since G is compact, let $T_c \subset G_c$ be the complexifications of the two groups. Then $X := G/T$ has a G-invariant complex structure, in fact several. These correspond to Borel subgroups $B_c \subset G_c$ which contain T_c. A Borel subgroup is (by definition) a maximal solvable closed subgroup, and it has a Levi decomposition $B_c = T_c N_c$. In the case $G = U(n)$, we can use the diagonal group for a maximal torus. The complexification of $U(n)$ is $GL(n, \mathbb{C})$, and a Borel subgroup is the upper triangular group. The complex structure comes from the fact that G/T embeds in G_c/B_c; in fact $G/T = G_c/B_c$. Thus an analytic character ξ of T gives rise to an equivariant complex line bundle $\mathcal{L}_\xi$ which corresponds to a complex character of B_c trivial on N_c. We can consider Dolbeault cohomology $H^{0,q}(X, \mathcal{L}_\xi)$. The action of G on X gives rise to representations on these cohomology groups that are computed by the Bott-Borel-Weil theorem. They are nonzero in at most one degree, and if so, the corresponding representation of G is irreducible. Conversely every irreducible representation can be realized this way, and one can say precisely for what (ξ, q). In particular it is possible to realize each representation using a ξ so that $\mathcal{L}_\xi$ has nontrivial holomorphic sections. Then the corresponding representation V_ξ is realized as holomorphic sections and ξ represents the *highest weight* of V_ξ.

The relation to induction from parabolic subgroups is as follows. The space of the induced module from ξ on $G \cap B_c = T$ to G consists of functions $f : G \to \mathbb{C}$ such that $f(gt) = \xi^{-1}(t) f(g)$. It is enough to restrict attention to C^∞ functions. This representation is too large, mainly because $N_c \cap G$ is trivial. To cut the space down, we observe that the action of $\mathfrak{g}$ as left invariant operators on functions on G complexifies to an action of $\mathfrak{g}_c$. So we require that the sections be annihilated by $\mathfrak{n}_c$. This condition is equivalent to requiring that the sections be holomorphic, same as that they be in the kernel of $\bar{\partial}$ for the complex structure coming from viewing G/T as G_c/B_c. When there are no holomorphic sections, we have to consider higher cohomology groups which are derived functors of global holomorphic sections.

In the case of a noncompact reductive group, it is not enough to consider finite dimensional representations. The systematic study of infinite dimensional representations of noncompact real reductive Lie groups was initiated by Harish-Chandra in the 1950's. His main goal was to prove a Plancherel formula. In this case, discrete series play the role of cuspidal representations. These are unitary irreducible representations that are characterized by the property that for any $v, w \in V$, the *matrix entry* $f_{v,w}(x) = \langle \pi(x)v, w\rangle$ is L^2. They exist only when G has a compact Cartan subgroup.

Harish-Chandra described the discrete series in terms of their distribution characters. Langlands conjectured that a similar result to the Borel-Bott-Weil theorem should hold, *i.e.* discrete series should be realized in Dolbeault cohomology, but one needs to impose extra square integrability conditions on the cycles and cocycles. This was achieved in [S]. The results extend to the case when T is replaced by a compact subgroup L which is the set of real points of a rational Levi component. They also suggest that one should attempt it for cases when L is no longer compact. This is not so easy. As before, there is a parabolic subalgebra Q_c that has a rational Levi component L_c and $L = Q_c \cap G$. If ρ is a representation of L, we can form a G-equivariant bundle $\mathcal{V}_\rho$. It will not have holomorphic sections unless L is compact. But ρ gives rise to a sheaf of germs of holomorphic sections $\mathcal{S}_\rho$ on X, and one can try to compute $H^{0,q}(X, \mathcal{S}_\rho)$. One complication is that G/L is only an open subset of G_c/Q_c. But much more serious is the problem that $\bar{\partial}$ might not have closed range, so it is not even clear how to define the cohomology.

One of the techniques that Harish-Chandra introduced was to reduce analytic problems about a representation of a Lie group to algebraic problems about representations of the Lie algebra. Let K be a maximal compact subgroup of G. If (π, V) is an (infinite dimensional) representation of a reductive Lie group, then the map $G \times V \to V$ is in general not smooth, so it is not possible to define an action of the Lie algebra. Harish-Chandra considered V_K, the set of vectors v such that $g \mapsto \pi(g)v$ is C^∞ and $\{\pi(K)v\}$ spans a finite dimensional subspace. Then V_K is a dense subspace on which $\mathfrak{g}_c$ and K act. If π is irreducible, then $\mathrm{Hom}_K(W, V_K)$ is finite dimensional for any irreducible representation W of K. The V_K serve as the model for the definition of *admissible* $(\mathfrak{g}_c, K)$*-modules* ([W] or Knapp-Vogan, chapter I). Langlands classified irreducible $(\mathfrak{g}_c, K)$-modules in the spirit outlined earlier: every irreducible $(\mathfrak{g}_c, K)$-module is realized as a quotient of a Harish-Chandra induced module from a discrete series.

Here is an example of how infinite dimensional representations arise naturally. Consider the problem of computing the cohomology groups $H^*(\Gamma, V)$, where V is a representation of the group Γ ([M], chapter IV). These cohomology groups are the derived functors of the functor of Γ-invariants in the category of Γ-modules. Suppose we are in the special case when Γ is a discrete torsion free subgroup of a connected semisimple group G. Recall that K is a maximal compact subgroup of G. The space $X = G/K$ is called a symmetric space, and Γ acts on the left. For example when $G = SL(2, \mathbb{R})$ and $K = SO(2)$, this is the upper half plane. Since X is contractible, in fact isomorphic to $\mathbb{R}^n$, the de Rham complex $A^*(X)$ is exact and therefore a free resolution. The group Γ acts on this complex because it acts on X on the left. So

$$H^*(\Gamma,\mathbb{C}) = H^*(A(X)^\Gamma).$$

This formula can be rewritten further as follows. Since the tangent space of G has a basis of left invariant vector fields, the de Rham complex for the group G can be written as

$$C^q(G;C^\infty(G)) := \mathrm{Hom}(\bigwedge^q \mathfrak{g}, C^\infty(G)).$$

Because Γ acts torsion free, $C^q(G;C^\infty(G))^\Gamma = \mathrm{Hom}(\bigwedge^q \mathfrak{g}, C^\infty(\Gamma\backslash G))$. This has a subcomplex $C^q(\mathfrak{g},K;C^\infty(\Gamma\backslash G))$ of forms which are annihilated by contraction with elements in the Lie algebra of K. Then the projection map $\pi : G \to X$ gives an isomorphism between this subcomplex and $A^q(X)^\Gamma$.

The above complex $C^q(\mathfrak{g},K,C^\infty(\Gamma\backslash G))$ can be constructed for any $(\mathfrak{g},K)$-module (V,π), not just $C^\infty(\Gamma\backslash G)$. Its cohomology groups are denoted by $H^*(\mathfrak{g},K;V)$, and they are called $(\mathfrak{g},K)$ cohomology. So we write

$$H^*(\Gamma) = H^*(\mathfrak{g},K;C^\infty(\Gamma\backslash G)).$$

Assume that $\Gamma\backslash G$ is also compact. Then a well known result of Gelfand-Piatetskii-Shapiro says that $L^2(\Gamma\backslash G)$ decomposes discretely,

$$L^2(\Gamma\backslash G) = \bigoplus m(\pi,\Gamma)V_\pi.$$

We can substitute $L^2(\Gamma\backslash G)^\infty$ for $C^\infty(\Gamma\backslash G)$ and take advantage of this decomposition. The end result is known as *Matsushima's formula*,

$$H^*(\Gamma) = \sum m(\pi,\Gamma)H^*(\mathfrak{g},K,V_\pi),$$

and I refer to [BW] for details. If G has no compact factors, the only finite dimensional (π,V_π) that can occur is the trivial representation. So to get information about $H^*(\Gamma)$, one needs to know how to compute $H^*(\mathfrak{g},K,V_\pi)$ for infinite dimensional unitary representations.

Cohomological induction can be thought of as an algebraic version of Dolbeault cohomology for $(\mathfrak{g}_c,K)$-modules. It was first introduced by Zuckerman in lectures at IAS in 1978 and independently by Parthasarathy in some special cases around the same time. It was motivated by the problem of constructing the $(\mathfrak{g}_c,K)$-modules corresponding to discrete series, and for computing $(\mathfrak{g},K)$ cohomology. Philosophically, the idea is to replace holomorphic sections of $\mathcal{S}_\rho$ by their germs at the identity in G/L, and to say that such a section is *global* if its K-translates generate a finite dimensional space. This can be phrased in a completely algebraic way in terms of K and the universal enveloping algebra $U(\mathfrak{g}_c)$ of $\mathfrak{g}_c$. Not only does this avoid the analytic difficulties, but properties of this construction follow from standard homological algebra.

Some of the problems of giving an analytical definition of Dolbeault cohomology were resolved only recently [Wo]. Even so, the fiber of the bundle $\mathcal{V}_\rho$ has to be finite dimensional, which is not sufficient for many applications. On the other hand, the algebraic construction has been used since 1978, and there are a number of important applications:

It is well suited for computing the $(\mathfrak{g},K)$ cohomology groups that appear in Matsushima's formula, [VZ] and [Ku].

It is possible to say a great deal about the K-structure of a cohomologically induced module. Standard cohomological algebra gives a formula for the restriction of a cohomologically induced module to the maximal compact subgroup K. In the case of discrete series this is called Blattner's formula.

It plays a central role in the classification of $(\mathfrak{g}_c, K)$-modules using *lowest K-types* due to Vogan, [V1]. In this classification each irreducible $(\mathfrak{g}_c, K)$-module is exhibited as a subquotient of a cohomologically induced module from a parabolic subgroup Q_c. The role of the discrete series is replaced by certain irreducible representations admitting *fine K-types.*

Given the relation to discrete series and the fact that the representations occurring in Matsushima's formula must all be unitary, it was conjectured from the onset that cohomological induction should preserve unitarity. The first general results of this nature were proved in [V2]. It is widely believed that the unitary dual is built up out of representations called *unipotent* via cohomological induction and complementary series. Such results are known for complex classical groups [Ba]. In this case cohomological induction coincides with Harish-Chandra induction, so this is not the main difficulty. The unitary groups form another special case; here the unipotent representations are themselves cohomologically induced from unitary characters. So there is a precise conjecture stating that an appropriate portion of the unitary dual of a unitary group is obtained by cohomological induction of unitary characters. In this case, the problem is mainly about cohomologically induced modules.

The original version of Zuckerman was never published. For a long time, the only account in textbook form was [V1]. The book by Knapp-Vogan is a complete treatment of the algebraic aspects of cohomological induction for real reductive groups, and it is readable by a student who has mastered a standard beginning level course in Lie groups, say the material in [H]. However, it is very technical and I believe intended more for a student who is already planning to do research in this area, so already familiar with the basics of the theory of infinite dimensional representations.

There is an extensive bibliography, historical notes and background material. The exposition is very clear and detailed, and there are appendices dealing with background material. For example I found the appendix on spectral sequences very useful for teaching this topic in a graduate course.

For the expert, this is a very comprehensive reference, and great care is taken to present the most general possible results. The only (very minor) objection might be that the exposition is too long at times. In particular, standard homological algebra arguments are often spelled out in too much detail.

Here are some of the topics and features that make this such a good reference:

The results are presented for more general cases than real points of linear connected reductive groups.

The equivalence of the Langlands and Vogan classifications of irreducible $(\mathfrak{g}_c, K)$-modules is treated in great detail.

There is an extensive treatment of the results on irreducibility and non-vanishing of cohomologically induced modules.

Yet another important feature is the departure from the original construction described in [V1]. Knapp-Vogan use a different definition of cohomological induction which is inspired by a functor introduced by Bernstein in the context of $\mathcal{D}$-modules called *fiber integration.* Recall $Q_c = L_c N_c$, where L_c is rational. The problem is to construct a $(\mathfrak{g}_c, K)$-module V from an $(\mathfrak{l}_c, L\cap K)$-module W. First one inflates W to $\widetilde{W}$, a representation of the Lie algebra $\mathfrak{q}_c$ of Q_c. Then one *induces* it up to $\mathfrak{g}_c$ by taking $U(\mathfrak{g}_c)\otimes_{U(\mathfrak{q}_c)}\widetilde{W}$. The result is a $(\mathfrak{g}_c, L\cap K)$-module. Let $R(\mathfrak{g}_c, K)$ be the algebra of left and right K-finite distributions on G supported on K, similarly $R(\mathfrak{g}_c, L\cap K)$. Then to construct a $(\mathfrak{g}_c, K)$-module from a $(\mathfrak{g}_c, L\cap K)$-module X, one takes $R(\mathfrak{g}_c, K)\otimes_{R(\mathfrak{g}_c, L\cap K)} X$. The composition of these functors is the version of cohomological induction used by Knapp-Vogan, and here is one advantage. For unitarity questions one needs to know what the hermitian dual of a cohomologically induced module is. This problem is awkward using the original definition, but was eventually settled in [EW]. Using the newer definition, one finds that there is another natural "*dual*" functor, namely the composition of $\mathrm{Hom}_{R(\mathfrak{g}_c, L\cap K)}[R(\mathfrak{g}_c, K), X]$ with $\mathrm{Hom}_{U(\mathfrak{q}_c)}[U(\mathfrak{g}_c), \widetilde{W}]$. The fact that these two functors are in duality comes down to a formal check that a diagram commutes. As a consequence one gets a formula for the hermitian dual of a cohomologically induced module in terms of these two constructions.

The material in Knapp-Vogan has a substantial overlap with [W]. Wallach's book seems more focused on the applications and also treats topics from harmonic analysis on reductive groups.

Even though this book presents such a complete view of the algebraic theory, the topic of cohomological induction is not exhausted; it is important and appealing from many different points of view. For the analytic version of this construction, in addition to [Wo], various problems are treated in work of Barchini, Knapp and Zierau ([BZ] and references therein). The geometric version of cohomological induction in the setting of $\mathcal{D}$-modules is in some sense more natural. But the results rely on much more sophisticated homological algebra. For example, the fiber integration functor mentioned earlier is not adequate. One needs to consider its derived functors, and it is not clear what the right category should be. I refer the interested reader to [BL], [HMSW] and [MP] for these developments.

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