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Local cohomology: An algebraic introduction with geometric applications, by M. P. Brodman and R. Y. Sharp, Cambridge University Press, 1998, xv+416 pp., \$69.95, ISBN 0-521-37286-0

The title refers to the local cohomology theory introduced by Grothendieck in his 1961 Harvard seminar [4] and his 1961–62 Paris seminar [5]. Since I attended the Harvard seminar and wrote the notes for it, local cohomology has always been one of my favorite subjects. To understand what it is, let us review a little the interactions between algebra and geometry that make up the subject of algebraic geometry.

Of course polynomial equations have always been used to define algebraic varieties, but it is only in the early 20th century with the rise of algebraic structures as objects of study that algebraic geometry and local algebra (the study of local rings) have become closely linked. Much of this connection is due to the work of Zariski, who discovered that a simple point on a variety corresponds to a regular local ring. He also explored the geometrical significance of integrally closed rings in relation to normal varieties.

Meanwhile, methods of homology and cohomology, which first appeared in topology, were making their way into algebra and geometry. The great work of Cartan and Eilenberg [2] formalized the techniques of cohomology into the abstract theory of homological algebra, including projective and injective resolutions, derived functors, and Ext and Tor. Auslander and Buchsbaum [1] introduced the notion of regular sequence and depth of a module (which they called codimension) and proved the fundamental result for a finitely generated module over a regular local ring, that its homological dimension plus its depth (the maximum length of a regular sequence) is equal to the dimension of the local ring.

On the other hand, the work of Kodaira and Spencer showed the usefulness of cohomology in the study of complex manifolds, and Serre in his famous paper FAC [6] showed that these same techniques could be extended to a theory of cohomology of coherent sheaves in abstract algebraic geometry. Grothendieck's extension of this theory to arbitrary schemes further strengthened the ties between algebra and geometry by making any ring A a subject for study in algebraic geometry via its geometrical reincarnation as Spec A.

The stage is now set for the introduction of local cohomology. The seeds were already present in FAC; only the definition was lacking. Here it is: Let X be a topological space, let Y be a closed subset, and let $\mathcal{F}$ be a sheaf of abelian groups on X. Define $\Gamma_Y(X, \mathcal{F})$ to be the group of those global sections of $\mathcal{F}$ that have support in Y. Define the local cohomology groups $H^i_Y(X, \mathcal{F})$ as the derived functors of the functor $\Gamma_Y(X, \cdot)$ on the category of abelian sheaves on X. As such, this is not a new concept, because the notion of cohomology of a sheaf with supports existed already. It is rather the uses to which it is put that make it a new theory.

The importance of local cohomology derives principally from two aspects. One is to create more flexibility in the calculation of cohomology of sheaves on schemes. There is a long exact sequence relating the cohomology of X, the local cohomology

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$H^i_Y(X, \cdot)$ and the cohomology of the open complement $U = X - Y$. There is also a sheaf version $\mathcal{H}^i_Y(\mathcal{F})$, which can be computed locally along Y and is linked to the global version by a spectral sequence. This allows a passage from local to global which was used effectively by Grothendieck, particularly in applications to Lefschetz-type theorems for the Picard group and the fundamental group of a variety.

The other important aspect of local cohomology is that it allows a translation of global results on projective space to purely algebraic results on the corresponding polynomial ring, and thence to its local ring at the origin. This generates local ring analogues of projective theorems, which in turn can be useful in local study of algebraic varieties.

Here are two typical results. If A is a noetherian local ring with maximal ideal $\mathfrak{m}$, and M an A-module, we denote the local cohomology groups $H^i_{V(\mathfrak{m})}(\mathrm{Spec}\ A, \tilde{M})$ by $H^i_{\mathfrak{m}}(M)$. The first result (foreshadowed by one of the last results in FAC, concerning "variétés k-fois de première espèce") is that the depth of M is equal to the least integer k for which $H^k_{\mathfrak{m}}(M)$ is non-zero. So the whole theory of depth can be re-established using local cohomology. The second is the local duality theorem, a local analogue of Serre duality for a projective variety, which says (under suitable hypotheses) that the local cohomology modules $H^i_{\mathfrak{m}}(M)$ are dual to $\mathrm{Ext}^{n-i}_A(M, \omega)$, where n is the dimension of A and ω is a dualizing module. This local duality theorem was an essential tool in Grothendieck's later generalized duality theorem for a projective morphism of schemes.

Since these ideas were introduced into algebraic geometry in the early 1960's, the methods of local cohomology have also been embraced by ring theorists, who greatly expanded and developed the algebraic aspects of the theory. However, writers of textbooks have given the theory only cursory mention, so that up to now there has been no source except for the original seminars of Grothendieck and research articles.

The book under review will surely be welcomed by algebraists, for it provides a complete and detailed introduction to the theory of local cohomology for rings and modules, together with applications, written entirely in the language of commutative algebra. Sheaves, topological spaces, and their cohomology are not mentioned until the very last section of the book, where, like a roman à clef, they reveal the secret meaning of earlier constructions.

For a ring A, an ideal $\mathfrak{a}$, and a module M, the local cohomology modules are defined as the derived functors, on the category of A-modules, of the functor $\Gamma_{\mathfrak{a}}(M)$, which is the module of those elements of M annihilated by any power of $\mathfrak{a}$. The book covers basic topics such as the relation to depth and the local duality theorem mentioned above, finiteness theorems, change of rings, the graded case, together with applications, which are local algebraic analogues of results in algebraic geometry: vanishing theorems, Castelnuovo-Mumford regularity, and connectivity theorems of Grothendieck, Fulton and Hansen.

The authors' decision to develop the subject purely algebraically has the obvious advantage that no knowledge of algebraic geometry is necessary to read the book. From my point of view, however, it sometimes results in obscuring the essential nature of the material. For example, in Chapter 2 they introduce the *ideal transform* of a module M with respect to an ideal $\mathfrak{a}$ to be

$$D_{\mathfrak{a}}(M) = \varinjlim_{n \in \mathbb{N}} \operatorname{Hom}_A(\mathfrak{a}^n, M) \ ,$$

and they spend considerable space developing its properties. Only at the end of the book do we learn that this is nothing else than the sections of the sheaf $\tilde{M}$ over the open set of $X = \operatorname{Spec} A$ obtained by removing the closed set $V(\mathfrak{a})$. From that point of view, the results of Chapter 2 are mostly trivial.

Sometimes the order of the material seems unnatural. For example, in Chapter 7 they prove that if M is a finitely generated module over a noetherian local ring $A, \mathfrak{m}$, then the local cohomology modules $H^i_{\mathfrak{m}}(M)$ are Artinian. This is a trivial consequence of the local duality theorem (proved only later in Chapter 11), but they don't say so. The only reason I can see for a separate proof is that they avoid the extra hypothesis that A be a quotient of a local Gorenstein ring, which is necessary for the duality theorem, but they don't say this either.

Also in Chapter 7 they introduce something called the secondary representation of an Artinian module, which seems quite mysterious until we learn later (Chapter 10) that it is just the Matlis dual of the usual primary decomposition.

I was pleased to see that the authors make such good use of an old example of mine that it appears seven times throughout the book, and even merits mention in the preface. But the geometry is lost. For me it is a very simple idea: just map the affine plane $\mathbb{A}^2$ into the affine four-space $\mathbb{A}^4$ in such a way that two points coincide but otherwise the map is injective. This creates an irreducible variety in $\mathbb{A}^4$ having a singularity that locally looks like two planes meeting. Its local ring has depth 1 instead of the expected depth 2, which creates all kinds of pathologies. None of this is explained in the book. Instead, they introduce the example (p.42) by saying, let V be the affine algebraic set in $\mathbb{A}^4$ given by

$$V : V_{\mathbb{A}^4}(X_1X_4 - X_2X_3,\ X_1^2X_3 + X_1X_2 - X_2^2,\ X_3^3 + X_3X_4 - X_4^2) \ .$$

It must take an amazing algebraic intuition (which I don't have) to visualize that. And, by the way, the figure on p.45 is incorrect: it shows the surface crossing itself along a line, when in fact it meets itself only in one point.

In conclusion, this book does a great service in making available in one place a wealth of material about local cohomology in a form accessible to those readers with little knowledge of algebraic geometry. The bibliography and index are excellent (though they unaccountably omitted Eisenbud's now standard book on commutative algebra [3]). It will surely become a basic reference book for the subject.

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