BOOK REVIEWS

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Character theory of finite groups, by Bertram Huppert, de Gruyter, Berlin, 1998, vi+618 pp., $168.95, ISBN 3-11-015421-8

The character theory of finite groups is the oldest part of what is now called "group representation theory", although when group characters were first discovered by G. Frobenius in 1896, they did not appear in their modern representation-theoretic context. Characters first arose as part of Frobenius' solution of a problem posed by R. Dedekind: factoring the so-called "group determinant". But it would be superfluous to give any further details here about group determinants or about other aspects of the early history of character theory because this material is beautifully presented by T. Y. Lam in his two-part historical survey [2] and [3].

According to the modern definitions (which are also due to Frobenius), a representation of a group G is a homomorphism D from G into the group $GL(n, K)$ of invertible $n \times n$ matrices over some field K. (The integer n is called the degree of the representation.) The character χ associated with a representation D is the function from G into K defined by $\chi(g) = \text{trace}(D(g))$. Classically, the field K was taken to be the complex numbers, and because this classical case is the primary focus of Huppert's book, we will limit our discussion to complex representations and characters in this review.

It is easy to see that the set of characters of G is closed under addition. Also, because the degree of a character χ (defined to be its value at the identity of G) is clearly equal to the degree of the corresponding representation, we see that $\chi(1)$ is always a positive integer. It follows that every character can be written as a sum of irreducible characters, which we can define to be those characters that cannot be decomposed as proper sums of characters. The set of irreducible characters of G is denoted $\text{Irr}(G)$, and this set is the fundamental object of study in this book. (Note that the full set of characters of G can be recovered as the set of all nonnegative integer linear combinations of $\text{Irr}(G)$.)

The characters of G are easily seen to be class functions, which means that they are constant on the conjugacy classes of G. It turns out that the set $\text{Irr}(G)$ is a basis for the full complex vector space of class functions of G, and in fact the irreducible characters form an orthonormal basis with respect to the Hermitian inner product $(\alpha, \beta)_G = (1/|G|) \sum \alpha(x)\overline{\beta(x)}$, where α and β are arbitrary class functions, and the sum runs over the elements $x \in G$. In particular, it follows that the number of irreducible characters of G is equal to the dimension of the space of

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class functions, and this, of course, is the number of conjugacy classes of G. Some of the other basic properties of characters that were discovered by Frobenius and that are proved in the first 8 of the 46 chapters of this book are the facts that the irreducible character degrees divide $|G|$, the sum of their squares is equal to $|G|$, and products of characters are always characters.

As is discussed in Lam's historical article [3], the really exciting developments in the early years of character theory were its applications to "pure" group theory. Most notable among these were the theorem of W. Burnside asserting that a group whose order is divisible by only two primes must be solvable and the theorem of Frobenius giving the structure of a transitive permutation group in which only the identity fixes as many as two points. These results and several other applications of the theory are presented in detail in Huppert's book, of course, along with a number of variations and extensions. We remark that Burnside's theorem can now be proved by a character-free (but quite subtle and difficult) argument, while Frobenius' theorem has resisted all attempts to find a character-free proof. (In addition to a full proof using characters, Huppert includes the "state of the art" in character-free partial proofs of Frobenius' theorem.)

The use of characters for proving "pure group theory" theorems has continued, of course, reaching its high-water mark in the Feit-Thompson theorem that groups of odd order are solvable. (The proof of this result is heavily dependent on both character theoretic and group theoretic techniques.) While Huppert includes a number of other significant group theory theorems, the main thrust of his book is in a different direction. Its principal focus is on the properties of characters themselves and on how character theoretic and group theoretic properties reflect each other, especially in groups (like solvable groups) with an abundance of normal subgroups. Also, the emphasis is on the characters of groups in general and not on the character theory of particular types of groups such as symmetric groups or groups of Lie type.

Chapters 19 through 22 are concerned with what is generally called "Clifford theory", which is the study of how the irreducible characters of a group interact with the normal subgroups. (These results stem from the seminal paper [1] of A. H. Clifford, but several other authors have also contributed significantly.) The basic idea here is that if $N \triangleleft G$, then G acts to permute the characters of N, and in particular G permutes $\mathrm{Irr}(N)$. Clifford showed that the restriction to N of an irreducible character of G is a multiple of the sum of the characters of some G-orbit on $\mathrm{Irr}(N)$. Using the "Clifford correspondence", one can reduce to the special case where the relevant G-orbit on $\mathrm{Irr}(N)$ consists of a single (invariant) irreducible character θ, and then one studies those irreducible characters of G whose restriction to N is some multiple $e\theta$ of θ. The integers e that arise in this way turn out to be the degrees of certain "projective representations" of the factor group G/N. (The theory of projective representations, which is presented in Chapter 20 of Huppert's book, antedates Clifford; it was founded by I. Schur, who was Frobenius' student.) It is especially important to understand when it is possible to take $e = 1$. Equivalently, given $\theta \in \mathrm{Irr}(N)$, one would like to have conditions sufficient to guarantee the existence of a character $\chi \in \mathrm{Irr}(G)$ such that θ is the restriction of χ to N. Huppert presents many of the known conditions of this type. (For example, if θ is invariant and the order and index of N are coprime, then θ extends to G.)

A major focus of Huppert's book is the question of what one can say about a group from a knowledge of the degrees of its irreducible characters. One can determine, for example, whether or not a group is nilpotent from what Huppert calls the degree pattern: the information giving the degrees and the number of irreducible characters of each degree. More remarkable, perhaps, is the fact that a great deal of information about G can be recovered from a knowledge of just the set $\mathrm{cd}(G)$ of degrees of the irreducible characters. For example, Huppert proves the result of J. Thompson asserting that if all members of $\mathrm{cd}(G)$ exceeding 1 are divisible by some fixed prime p, then G has a normal p-complement. He also presents the theorem of the reviewer and D. S. Passman that if every member of $\mathrm{cd}(G)$ exceeding 1 is prime, then G is solvable and $|\mathrm{cd}(G)| \leq 3$.

In fact, just the knowledge of the cardinality $|\mathrm{cd}(G)|$, is sufficient to determine important structural information about G. For example, if $|\mathrm{cd}(G)| \leq 3$, then G must be solvable, and its derived length $\mathrm{dl}(G)$ is at most equal to $|\mathrm{cd}(G)|$. If $|\mathrm{cd}(G)| \geq 4$, then G need not be solvable; but if one assumes that G is solvable, it is conjectured that the so-called Taketa inequality $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ remains valid. (By a result of K. Taketa, presented in Chapter 24, this inequality is known to be valid for a large class of solvable groups called "M-groups". These include all groups of prime-power order.)

Actually, only a few examples of solvable groups where the derived length is at least 4 and is as large as $|\mathrm{cd}(G)|$ are known, and Huppert presents all of the known examples of this phenomenon. (In fact, one of the strengths of this book is the great wealth of examples it contains.) No examples are known where the derived length of G exceeds 5 and is as large as $|\mathrm{cd}(G)|$, and it appears, therefore, that the conjectured Taketa inequality is too weak. In the case of groups of prime-power order (where the Taketa inequality is known to be valid) it appears that the correct upper bound on the derived length should be logarithmic in $|\mathrm{cd}(G)|$. In Chapter 26, for example, Huppert discusses several infinite families of such groups for which he computes derived lengths and character degrees, and he observes that a logarithmic bound holds in each case.

In recent decades, a coherent theory of characters of solvable groups has arisen, with applications to M-groups and to various other problems. (Instrumental in this development have been E. C. Dade, the reviewer, and others.) One aspect of this solvable character theory is what can be called π-theory, where π represents a set of primes. Huppert devotes his Chapter 40 to an introduction to π-theory, and one of the applications (in Chapter 41) is a proof of the solvable case of a conjecture of W. Feit. (The first proof in this solvable case was given by G. Amit and D. Chillag.) If $\chi \in \mathrm{Irr}(G)$, we write $f(\chi)$ to denote the smallest positive integer f such that the values of χ all lie in the cyclotomic field of fth roots of unity. (It is easy to see that $f(\chi)$ always divides $|G|$.) Feit's conjecture is that G must contain an element of order $f(\chi)$ for every character $\chi \in \mathrm{Irr}(G)$.

Huppert also discusses several "classical" topics without which a character theory book would have to be considered incomplete. For example, in Chapter 34 he presents the powerful theorem of R. Brauer (a student of Schur) that can be used to prove that a class function φ of a group G is actually a character. Brauer's result is that if the restriction of φ to every nilpotent subgroup E of G is a difference of characters of E, then φ must be a difference of characters of G. (Additional information may then be available to show that φ is actually a character.) Applications of Brauer's theorem are presented in Chapter 35. Typical among these is the result that if the integer $|G|/\chi(1)$ is not divisible by the prime p, where $\chi \in \mathrm{Irr}(G)$, then $\chi(x) = 0$ for every element $x \in G$ of order divisible by p. The clever proof proceeds by creating a "new" class function φ, vanishing on elements of order divisible by p and agreeing with χ on the rest of G. Brauer's theorem can be used to prove that φ is a difference of characters, and then an easy argument with inner products shows that $\varphi = \chi$, as desired.

Huppert has a well established reputation as a superb expositor of group theory, and this book clearly follows in the tradition of his earlier work. Like his previous books, this one is very well written. Also, it is filled with examples, and it has an extremely wide selection of topics, many of which have never before appeared in book form. Another strength of this book is the collection of remarks in many of the chapters. A number of results that are not presented in detail because their proofs are too technical or too long are at least described, and references are given.

While Huppert's book certainly belongs on the shelves of all those interested in group representation theory, it should prove especially invaluable to graduate students who are studying or doing research in character theory. It is a welcome addition to the literature.

References

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