BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 3, Pages 351–357
S 0273-0979(00)00866-1
Article electronically published on March 2, 2000

Tame topology and o-minimal structures, by Lou van den Dries, Cambridge Univ. Press, New York, 1998, x + 180 pp., \$39.95, ISBN 0-521-59838-9

§1 Semialgebraic and subanalytic geometry

In classical algebraic geometry over an algebraically closed field, when an affine algebraic variety is projected to a lower dimensional affine space, the image is a finite boolean combination of algebraic varieties. The situation is more complicated when studying algebraic varieties over the real field. For example, when the circle $x^2 + y^2 = 1$ is projected on x-axis the image is the closed interval $[-1, 1]$. Thus when studying solutions to polynomial equations over the reals, we must also study polynomial inequalities.

We say that a subset of $\mathbf{R}^n$ is *semialgebraic* if it is a finite Boolean combination of solution sets to polynomial equations

$$p(X_1, \dots, X_n) = 0$$

and polynomial inequalities

$$q(X_1, \dots, X_n) > 0.$$

It is easy to see that any semialgebraic set arises as the projection of an algebraic variety in a higher dimensional real affine space. One might worry that even more complicated sets could arise when we project semialgebraic sets, but this does not occur. Indeed, the Tarski-Seidenberg Principle guarantees that if $X \subseteq \mathbf{R}^n$ is semialgebraic and $f : \mathbf{R}^n \to \mathbf{R}^m$ is a polynomial map, then the image of X under f is semialgebraic. One consequence of the Tarski-Seidenberg Principle is that the closure of a semialgebraic set is semialgebraic.

Our hopes for understanding real algebraic varieties would be futile if semialgebraic sets could be wild, but fortunately semialgebraic sets have many tame topological and geometric properties. For example:

• Stratification: If X is semialgebraic, then there are finitely many disjoint semialgebraic sets $X_1, \dots, X_n$ such that $X = X_1 \cup \dots \cup X_m$; each X_i is a connected real analytic manifold; and if $\overline{X}_i \cap X_j \neq \emptyset$ for $i \neq j$, where $\overline{X}_i$ is the closure of X_i, then $\overline{X}_i \supseteq X_j$ and $\dim X_i > \dim X_j$. In particular, any semialgebraic set has finitely many connected components, and the boundary of a semialgebraic set is a semialgebraic set of lower dimension.

• Piecewise smoothness of semialgebraic maps: We say that a function is semialgebraic if its graph is semialgebraic. If $f : X \to \mathbf{R}$ is semialgebraic, then X can be partitioned into finitely many disjoint semialgebraic sets $X_1, \dots, X_n$ such that each restriction $f|X_i$ is analytic.

• Triangulation: Any compact semialgebraic set admits a semialgebraic triangulation.

• Finiteness of topological type: Suppose $X \subseteq \mathbf{R}^{n+m}$ is semialgebraic. Let $X_a = \{x \in \mathbf{R}^n : (x, a) \in X\}$ for $a \in \mathbf{R}^m$. We call X a *semialgebraic family* of semialgebraic sets if there are $a_1, \dots, a_l \in \mathbf{R}^m$ such that for all $a \in \mathbf{R}^m$ there is $i \leq l$ and a semialgebraic homeomorphism $f : X_a \to X_{a_i}$. In other words, a

2000 *Mathematics Subject Classification.* Primary 03C64; Secondary 14P10, 14P15.

semialgebraic family represents only finitely many semialgebraic homeomorphism types.

The results above are discussed in the excellent texts [1] and [3].

It is tempting to look for larger classes of subsets of $\mathbf{R}^n$ sharing the tame properties of the semialgebraic sets. One natural generalization is the subanalytic sets. Let M be a real analytic manifold. We say that $X \subseteq M$ is *semianalytic* if for all $a \in M$ there is an open neighborhood U of a such that $X \cap U$ is a finite union of sets

$$\{x \in U : f_1(x) = \cdots = f_m(x) = 0, g_1(x) > 0, \ldots, g_l(x) > 0\}$$

where $f_1, \ldots, f_m, g_1, \ldots, g_l$ are analytic on U.

Locally semianalytic sets behave much like semialgebraic sets. For example, any semianalytic $X \subseteq M$ with compact closure is the union of finitely many semianalytic sets each of which is a real analytic manifold. Unfortunately semianalytic sets are not stable under projection. A trivial example is the unbounded semianalytic set

$$(*) \qquad \{(\frac{1}{n}, n) : n = 1, 2, \ldots\}$$

since the projection to $\mathbf{R}$ is not semianalytic near 0. Even bounded semianalytic sets need not have semianalytic projections. The set

$$Y = \{(x, y, z, w) : 0 < x, y, w \leq 1, wy = x, \text{ and } z = ye^w\}$$

is semianalytic, but Osgood showed that the projection $\{(x, y, z) : \exists w\ (x, y, z, w) \in Y\}$ is not. To avoid this problem we look at the class of subanalytic sets introduced by Lojasiewicz and Hironaka. We say that $X \subseteq M$ is *subanalytic* if for every $a \in M$ there is an open neighborhood U of a and a semianalytic $Y \subseteq M \times \mathbf{R}^m$ with compact closure, such that $U \cap X$ is the projection of Y to M.

Subanalytic sets with compact closures exhibit all of the tameness properties of semialgebraic sets described above. Indeed, subanalytic sets in general exhibit these properties locally. The article [2] is an excellent introduction to subanalytic geometry. Unfortunately, sets like the graph of $e^{-\frac{1}{x^2}}$ are not subanalytic.

This leads us to a natural challenge from Grothendieck's *Esquisse d'un Programme* (see [10], [17], [19]):

Investigate classes of sets with the tame topological properties of semialgebraic sets.

O-minimality is the model theoretic response to this challenge.

§2 O-minimality

Although the notion of o-minimality arose first in model theory, a branch of mathematical logic, very few logical concepts are needed to appreciate, or prove, the basic results. An *ordered structure* on $\mathbf{R}$ is a sequence $\mathcal{S} = (S_1, S_2, \ldots)$ where S_n is a Boolean algebra of subsets of $\mathbf{R}^n$ with the following properties:

i) $\emptyset \in S_n$, $\mathbf{R}^n \in S_n$.
ii) The set $\{(x, y) : x, y \in \mathbf{R}^n \text{ and } x = y\} \in S_{2n}$.
iii) If $a \in \mathbf{R}$, then $\{a\} \in S_1$.
iv) The set $\{(x, y) : x, y \in \mathbf{R} \text{ and } x < y\} \in S_2$.
v) If $A \in S_n$, then $A \times \mathbf{R}$ and $\mathbf{R} \times A \in S_{n+1}$.
vi) If $A \in S_{n+1}$ and $B \subseteq \mathbf{R}^n$ is the projection of A onto the first n-coordinates, then $B \in S_n$.

If $A \in S_n$, we say that A is a *definable* set in the structure $\mathcal{S}$.

For example, if we take S_n to be the semialgebraic subsets of $\mathbf{R}^n$, then $\mathcal{S}$ is an ordered structure on $\mathbf{R}$. Taking S_n to be the subanalytic sets will not work because the unbounded set (*) above does not have a subanalytic projection.

Often we specify a structure by considering the smallest structure where certain sets are definable; we start with B_n, a collection of subsets of $\mathbf{R}^n$ for $n = 1, 2, \ldots$, and look for the smallest structure $\mathcal{S}$ such that $B_n \subseteq S_n$ for all n.[1]

Here are four interesting examples:

• $\mathcal{S}_{\mathrm{lin}}$, the semilinear sets, the smallest ordered structure containing all sets of the form

$$\{x \in \mathbf{R}^n : \sum r_i x_i = 0\}$$

for $r_1, \ldots, r_n \in \mathbf{R}$;

• $\mathcal{S}_{\exp}$, the smallest structure containing the semialgebraic sets and the graph of $x \mapsto e^x$;

• $\mathcal{S}_{\mathrm{an}}$, the smallest structure containing the semialgebraic sets and all subanalytic sets with compact closure;

• $\mathcal{S}_{\mathbf{Z}}$, the smallest structure containing the semialgebraic sets with the set of integers $\mathbf{Z} \in S_1$.

We would like to know which structures have "tame" definable sets. While this is a vague question, we can give a negative answer for $\mathcal{S}_{\mathbf{Z}}$. Every Borel subset of $\mathbf{R}^n$ is in $\mathcal{S}_{\mathbf{Z}}$, and questions as basic as whether every definable set is Lebesgue measurable depend on set-theoretic assumptions. Thus even simple-looking structures can have complicated definable sets.

In the early 80's van den Dries, Knight, Pillay and Steinhorn ([4], [16], [12]) isolated one property of ordered structures that leads to tame behavior. We say that an ordered structure $\mathcal{S}$ on $\mathbf{R}$ is *o-minimal* if every $A \in S_1$ is a finite union of points and intervals with endpoints in $\mathbf{R} \cup \{\pm\infty\}$. The semialgebraic sets are an o-minimal structure on $\mathbf{R}$. Since every semilinear set is semialgebraic, the semilinear sets also yield an o-minimal structure (with fewer definable sets).

Remarkably, although o-minimality only refers to subsets of the real line, it also has strong consequences for subsets of $\mathbf{R}^n$. Let $\mathcal{S}$ be an o-minimal structure. We inductively define the collection of *cells* as follows:

i) $X \subseteq \mathbf{R}$ is a cell if and only if it is either a point or an interval.

ii) If $X \subseteq \mathbf{R}^n$ is a cell and $f : X \to \mathbf{R}$ is a continuous definable function, then the graph of f is a cell.

iii) If $X \subseteq \mathbf{R}^n$ is a cell and $f, g : X \to \mathbf{R}$ are continuous definable functions and $f(x) > g(x)$ for all $x \in X$, then $\{(x, y) : x \in X \text{ and } f(x) > y > g(x)\}$ is a cell, as are $\{(x, y) : x \in X \text{ and } f(x) > y\}$ and $\{(x, y) : x \in X \text{ and } y > f(x)\}$.

Every cell is definable and definably homeomorphic to a product of intervals. If $\mathcal{S}$ contains the semialgebraic sets, then every cell is definably homeomorphic to $\mathbf{R}^d$ for some d. We can now state the fundamental theorem about o-minimal structures:

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[1] The definable sets are exactly the sets defined by first order formulas where we allow basic formulas $R_A(x_1, \ldots, x_n)$ asserting $(x_1, \ldots, x_n) \in A$ for each $A \in B_n$ and $n = 1, 2, \ldots$. While the definition we have given avoids the logical formalism, we lose the uniformity available when we fix a formula and vary the structure, one of the basic tools of model theory.

Cell Decomposition Theorem. *Let $\mathcal{S}$ be o-minimal.*

i) Every definable set can be partitioned into finitely many disjoint cells.

ii) If $f : X \to \mathbf{R}$ is a definable function, then there is a partition of X into finitely many cells, such that f is continuous on each cell. If $\mathcal{S}$ contains the semialgebraic sets, then for each m we can choose the partition such that f is $\mathcal{C}^m$ on each cell.

This is just the beginning. Suppose $\mathcal{S}$ is an o-minimal structure containing, at least, the semialgebraic sets.[2]

• Stratification: If X is definable and $p > 0$, then there are finitely many disjoint definable sets $X_1, \ldots, X_n$ such that $X = X_1 \cup \cdots \cup X_m$; each X_i is a connected $\mathcal{C}^p$ manifold; and if $\overline{X}_i \cap X_j \neq \emptyset$ for $i \neq j$, where $\overline{X}_i$ is the closure of X_i, then $\overline{X}_i \supseteq X_j$ and $\dim X_i > \dim X_j$. In particular, any definable set has finitely many connected components, and the boundary of a definable set is a definable set of lower dimension.

Though in general the partition may depend on p, in many important examples there is in fact a partition into analytic manifolds.

• Triangulation: Any compact definable set admits a definable triangulation.

• Any definable family represents only finitely many definable homeomorphism types.

• Curve selection: If $\mathcal{S} \supseteq \mathcal{S}_{\text{lin}}$, $X \subseteq \mathbf{R}^n$ is definable, and a is in the closure of X, then there is a continuous definable $f : (0,1) \to X$ such that

$$\lim_{x \to 1} f(x) = a.$$

• (Pillay [15]) If G is a definable group, then G is definably isomorphic to a Lie group.

§3 New examples

While the results above provide ample evidence for the claim that o-minimal structures are tame, this would be a rather sterile subject if we did not also have interesting new o-minimal examples. The first example beyond the semialgebraic sets is $\mathcal{S}_{\text{an}}$. Van den Dries [5] showed that $X \subseteq \mathbf{R}^n$ is definable in $\mathcal{S}_{\text{an}}$ if and only if the image of X under Φ is subanalytic where $\Phi : \mathbf{R}^n \to (0,1)^n$ is a semialgebraic bijection. In particular the bounded definable sets are exactly the subanalytic sets and $\mathcal{S}_{\text{an}}$ is o-minimal. Thus o-minimality explains much of the tame behavior of subanalytic sets.

The main breakthrough in the subject was Wilkie's result [20] that $\mathcal{S}_{\text{exp}}$ is o-minimal. Wilkie proceeded by giving a normal form for the definable sets. We say that $V \subseteq \mathbf{R}^n$ is an *exponential variety* if V is a solutions set to a system of exponential-polynomial equations like

$$e^{e^{y^2+1}} - ze^{x-2y} + 3zy = 7.$$

Model Completeness of Exponentiation. If $X \subseteq \mathbf{R}^n$ is definable in $\mathcal{S}_{\text{exp}}$, then there is an exponential variety $V \subseteq \mathbf{R}^{n+m}$ such that X is the projection of V to $\mathbf{R}^n$.

The topology of exponential varieties had previously been studied by Khovanskii [11], who showed that any exponential variety has finitely many connected components. If $X \subseteq \mathbf{R}$ is definable, then X is the projection of finitely many connected

[2] Many results can be proved without the assumption that $\mathcal{S}$ contains the semialgebraic sets.

sets. It follows that X is a finite union of points and intervals. Thus $\mathcal{S}_{\exp}$ is o-minimal, and all the definable sets are tame.

We give one striking application to Khovanskii's theory of "fewnomials". Let $\mathcal{F}_{n,m}$ be the collection of polynomials in $\mathbf{R}[X_1,\ldots,X_n]$ with at most m monomials. For $p \in \mathcal{F}_{n,m}$ let

$$V^+(p) = \{x = (x_1,\ldots,x_n) \in \mathbf{R}^n : \bigwedge_{i=1}^{n} x_i \geq 0 \wedge p(x) = 0\}.$$

We claim that there are only finitely many homeomorphism types of $V^+(p)$ for $p \in \mathcal{F}_{n,m}$. Let $\Phi_{m,n}(x_1,\ldots,x_n,r_{1,1}\ldots,r_{1,n},\ldots,r_{m,1},\ldots,r_{m,n},a_1,\ldots,a_m)$ be the formula

$$\exists w_1\ldots,w_m \ ((\bigwedge_{i=1^m} e^{w_i} = x_i) \wedge \sum_{i=1}^{m} a_i \prod_{j=1}^{n} e^{w_i r_{i,j}} = 0).$$

We see that Φ expresses

$$\sum_{i=1}^{m} a_i \prod_{j=1}^{n} x_j^{r_{i,j}} = 0.$$

Let $X_{\overline{r},\overline{a}}$ denote the set of $x \in \mathbf{R}^n$ such that $\Phi(x,\overline{r},\overline{a})$ holds. By o-minimality, $\{X_{\overline{r},\overline{a}} : \overline{r} \in \mathbf{R}^{mn}, \overline{a} \in \mathbf{R}^m\}$ represents only finitely many homeomorphism types.

A major direction for current research is to find even richer o-minimal structures. Here are a few important results.

• [6] $\mathcal{S}_{\mathrm{an,exp}}$ the smallest structure containing $\mathcal{S}_{\exp}$ and $\mathcal{S}_{\mathrm{an}}$ is o-minimal. This result is greatly extended in [8] and [9], where van den Dries and Speissegger work with functions given by either convergent generalized power series with positive real exponents or multisummable series instead of analytic functions.

• Wilkie [21] showed that the expansion of the real field by all Pfaffian functions is o-minimal. Speissegger [18] extended this by showing that the "Pfaffian closure" of an o-minimal structure is o-minimal.

§4 The book

Van den Dries' *Tame topology and O-minimal structures* can be viewed as a compelling argument that o-minimal structures are Grothendieck's desired framework for developing tame mathematics. Starting from scratch, van den Dries introduces o-minimal structures and carefully develops all of the key topological results about definable sets, including all of the results stated in §2 above, with the exception of Pillay's theorem on definable groups. Many of these results appear here for the first time, having only appeared before in the author's unpublished notes. In addition to results already mentioned, van den Dries treats Euler characteristic for definable sets, gives a clear treatment of Laskowski's proof of the Vapnik-Chervonenkis Property and begins to develop the notion of *definable spaces*, a topic that leads to the coordinate free approach to o-minimality developed in [7].

This long-awaited book will be indispensable to any student or researcher interested in o-minimal structures. It is written with remarkable precision and clarity and assumes only the most modest prerequisites from point-set topology and analysis. In particular, no knowledge of logic or model theory is assumed.

Let me conclude with a few comments about what is not in the book. If one can find any fault with the book, it is that van den Dries does not prove the o-minimality of any structure beyond the semialgebraic sets. There are a number of tantalizing hints of a "next volume" that would include proofs of the o-minimality of $\mathcal{S}_{\mathrm{an}}$ and $\mathcal{S}_{\exp}$, and the reviewer hopes that this plan is pursued.

Another interesting direction beyond the scope of the book is the more model theoretic investigation of o-minimal structures. This has been an exciting area of research, beginning with the work of Pillay and Steinhorn, through the recent striking trichotomy theorem of Peterzil and Starchenko [13]. Though the motivation comes from pure model theory, this work has already found interesting applications in the study of simple algebraic and semialgebraic groups over real closed fields [14].

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