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*Bernhard Riemann, 1826–1866: Turning points in the conception of mathematics*, by Detlef Laugwitz (translated by Abe Shenitzer), Birkhäuser, Boston, 1999, xvi + 357 pp., $79.50, ISBN 0-8176-4040-1

## 1. Introduction

The work of Riemann is a source of perennial fascination for mathematicians, physicists, and historians and philosophers of these subjects. He lived at a time when a considerable body of knowledge had been established, and his work changed forever the way mathematicians and physicists thought about their subjects. To tell this story and get it right, describing what came before Riemann, what came after, and the extent to which he was responsible for the difference between the two, is a task that the author rightly describes as both tempting and daunting. Fortunately he has succeeded admirably in this task, stating the technical details clearly and correctly while writing an engaging and readable account of Riemann's life and work. The word "engaging" is used very deliberately here. Any reader of this book with even a passing interest in the history or philosophy of mathematics is certain to become engaged in a mental conversation with the author, as the reviewer was. The temptation to join in the debate is so strong that it apparently also lured the translator, whose sensitive use of language has once again shown why he is much in demand as a translator of mathematical works.

The areas that Riemann dealt with, such as algebraic function theory, complex analysis, differential geometry, topology, trigonometric series, and mathematical physics, were so fundamental, and his approach to them was so original, that one cannot help being intrigued by the story. The author's presentation of this story is infused with his own views on the development of mathematics in the late nineteenth century, which frame the exposition, especially in the final chapter of the book. It would be unfair to attempt to summarize the author's views, which require a leisurely exposition in more space than can be afforded in a review. That being said, one point may perhaps be noted. The author cites with approval I. Grattan–Guinness' insight that geometric, analytic, and algebraic approaches co-existed in the analysis of the early nineteenth century, as exemplified by the approaches to integration. That description is certainly accurate. There was, nevertheless, a tendency for algebraic arguments to be regarded as more reliable, and the author is on very solid ground in asserting (p. 52) that every generation of mathematicians from the time of Leibniz on made some attempt to reduce analysis to algebra. Lagrange is a famous example, but Weierstrass also wrote about the importance of algebra. Near the end of his life, when his lectures on abelian functions were being printed as Vol. 4 of his collected works, he recalled that Riemann's solution of the Jacobi inversion problem had caused him to withdraw his own paper on the same subject and think further about the matter. He noted that the proof of the consistency of his results with those of Riemann, "demanded investigations of

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a fundamentally algebraic nature." General histories of algebraic function theory have recognized three different approaches to the subject in the nineteenth century, called arithmetical (Kronecker, Weierstrass, Dedekind, Weber), algebraic-geometric (Clebsch, Gordan, Brill, M. Noether, Severi), and transcendental (Abel, Riemann).

## 2. Complex analysis

Riemann stood athwart this path of algebraization, and that is what makes him appear to inhabit an entirely different universe, not only from the British mathematicians of the time, but even from Germans like Dedekind, who were close to him. Although he was considered a successor of Abel's work, his approach to the study of algebraic functions seems to stand like a lone tree in the midst of a vast plain when compared with the work leading up to it by Abel, Jacobi, Hermite, Göpel, and Rosenhain. It is well-known that, after Göpel and Rosenhain had introduced theta functions of two variables, Riemann and Weierstrass independently solved the Jacobi inversion problem. As Weierstrass noted, however, their approaches were so different that he withdrew his paper from publication and devoted himself to a long period of algebraic investigation in order to assimilate Riemann's work and translate it into the algebraic language that he favored. Comparing the two approaches, one gets a sense of the profound consequences that can result from a very slight shift in one's point of view. In his lectures on the subject of Abelian functions (volume 4 of his collected works) Weierstrass based the theory on the concept of a *Gebilde*, which is simply the subset of the space $\mathbb{C}^2$ satisfying a polynomial equation $p(z, w) = 0$, supplemented by a few ideal points corresponding to cases in which the equation is satisfied by what we would now call the point at infinity in the extended complex plane. To use the *Gebilde* Weierstrass required representations of the variables in a neighborhood of each point as power series in a parameter that could be thought of as a root of one of the variables. This approach reminds one of an algebraic variety, but Weierstrass requires a great many formulas to get to his results. The comparative simplicity of Riemann's approach through the Riemann surface no doubt explains why his method gives better insight into the problem. The author makes this point quite effectively in Chapter 1, showing how Riemann's approach brings out the double periodicity of the inverse of an elliptic integral. The author also points out (p. 92) that Puiseux also had the idea of expanding a function at a branch point in a series of fractional powers of the variable. In fact, as Riemann undoubtedly knew, Puiseux had the idea of a multi-sheeted surface to represent a multi-valued function, but he clung to a notation apparently due to Cauchy in which subscripts are used to keep the sheets separate, rather than fusing them over a branch cut. Again, small differences in one's point of view are important.

As already stated, in this approach to function theory, Riemann was sailing against a rather strong "algebraic" headwind. Weierstrass, though duly impressed with Riemann's genius, never quite approved of what he regarded as Riemann's lack of rigor. One historian of mathematics (unfortunately I do not remember who) once pointed out to me that Weierstrass' references to Riemann in his letters nearly always involve some mathematical criticism, indicative perhaps of a latent jealousy.

## 3. Geometry, physics, philosophy

The author has made the sound decision to treat Riemann's work on geometry, physics, and philosophy together in one chapter. Interestingly, while the name Riemann suggests (at least to the reviewer) a space whose metric is given by a positive-definite quadratic form, Riemann regarded this particular form as merely the simplest specific example of a more general object. In contrast, Einstein was guided in deriving his gravitational law by the desire to fulfill certain conditions using no derivatives of order higher than the second. (Like Riemann, he also recognized that adjusting the law to accommodate more precise observations might require more complicated expressions.) Riemann's work in this area was a natural continuation of Gauss' research into surfaces, but it was shaped by the important insight that the notion of distance has to be built up from the infinitesimal level to the finite, exactly opposite to the point of view that attempts to "tame" infinitesimals by defining them as limits of finite quantities. The most amusing pages in the book occur in this chapter, as the author tells how the aged Felix Klein strained language to make special relativity fit into his Erlangen Program. (The author notes Hans Wussing's 1984 book *The Genesis of the Abstract Group Concept*, in which this point is also made.) He points out the parallel between Klein's attempted classification and the twentieth-century systematization of Bourbaki, wisely noting, "We realize the fruitfulness of bringing conceptually distinct things under one roof but also the dangers of dogmatism."

## 4. Real analysis

Riemann's contribution to real analysis is less spectacular than his work in geometry, physics, and complex analysis. The bulk of it is contained in his posthumously published paper on the uniqueness of the representation of a function by trigonometric series. This work shows the strong influence of Dirichlet, who proposed the topic to Riemann. It is the only paper by Riemann in which sources are meticulously cited. (As we know from a letter he wrote to his father, Riemann had been given all these references by Dirichlet.) It contains, as a tool, the development of the Riemann integral, as distinguished from that of Cauchy. The author correctly points out that Cantor was led to infinite ordinal numbers and, indeed, to set theory in general through the problem of describing the geometric figures (sets) having the property that the only trigonometric series converging to zero on the figure is the one all of whose coefficients are zero. Strictly speaking, the focus of attention was on the complement of the figure. With convoluted syntax, a set $A$ is called a set of uniqueness if the only trigonometric series converging to zero on the *complement* of $A$ is the series all of whose coefficients are zero. Notice that one is nearly *forced* to use the language of sets to discuss this theorem. The deliberate use of the word *figure* above seems quite incongruous. (How *did* mathematicians manage to discuss their subject before set theory?)

As the author further points out, Cantor's transfinite ordinals, the basis of transfinite induction, have been largely replaced by other equivalent forms of the axiom of choice such as Zermelo's well-ordering principle (itself virtually identical to the principle of transfinite induction), the Hausdorff maximal principle, or its corollary, known as Zorn's lemma (first stated explicitly by Bochner). On this basis the author takes the point of view that set theory as it has come down to the present is mostly due to Dedekind and Hilbert. As he says (p. 322),

> ... from the viewpoint of our subject, Cantor is far less important than are Dedekind and Hilbert. Besides, Dedekind developed the set-based *mode of thought* before Cantor. Specifically, he formed subsystems with definite properties (cuts, ideals) of concretely given systems ... Hilbert started with unstructured sets without properties and imposed structures on them.

While granting the accuracy of the facts presented in this quotation, the reviewer would like to point out some other important aspects of Cantor's work, namely Cantor's characterization of the relation between points and sets, isolating the concept of the derived set of a set. This concept came to his attention because Riemann showed (Riemann's first theorem) that the auxiliary function $F(x)$ he had constructed to study a trigonometric series cannot have a corner, that is, different one-sided derivatives at a point. As a consequence, if this function is piecewise-linear, it must be linear, from which it follows that finite sets are sets of uniqueness. Because of the Bolzano–Weierstrass theorem, infinite sets of points (inside a fundamental period) at which a trigonometric series diverges must have points of accumulation. However, the no-corners principle still applies, showing that a set having only a finite number of points of accumulation is still a set of uniqueness. At that point, one is naturally led to consider the $n$th derived set, which, as the author points out, was the original source of ordinal numbers. As long as some derived set of finite order is finite, the set will be a set of uniqueness. And, although it is true that Zorn's lemma is much more commonly used than transfinite induction, the latter is not entirely forgotten. For example, most people who have studied measure theory have encountered the construction of the sigma-ring generated by a class of sets, a characterization that depends on transfinite ordinals. Likewise, some of the important counterexamples in both measure theory and point-set topology depend on transfinite ordinal numbers and the continuum hypothesis. If those were the only examples, the author's basic point could be considered as established. Indeed, in the opinion of the reviewer, Cantor's work would have flopped if not for the profound applications of it given by Borel, Lebesgue, and Baire, who created both metric set theory (measure theory) and descriptive set theory. The latter classifies sets according to their topological complexity and is a direct application of Cantor's derived sets, the concept that led to infinite ordinal numbers in the first place. Descriptive set theory has recently returned to its roots in the theory of uniqueness of trigonometric series and provided metatheorems showing that many of the known sufficient conditions for uniqueness cannot be necessary. Cantor's work lives on.

## 5. Conclusion

The format of the book is excellent, especially the plentiful supply of photographs of people and places. An unusual feature of this book is that, while the entire work is a translation from German, quotations in German are both translated and displayed in the original. This feature makes the book a few pages longer than it would have been otherwise, but will probably be appreciated by historians interested in the exact words used to support an argument. The book will serve as an interesting read and also a useful reference. It is highly recommended.

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