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Extending H_∞ control to nonlinear systems, by J. W. Helton and M. R. James, Frontiers in Applied Mathematics, SIAM, 1999, xiv+332 pp., $59.50, ISBN 0-89871-440-0

A fundamental problem in control theory is to design controllers which give satisfactory performance in the presence of uncertainties such as unknown model parameters and disturbances which enter the system dynamics. H_∞ control theory originated in an effort to codify classical control methods where frequency response functions are shaped to meet certain performance objectives. Linear H_∞ control theory has developed extensively since the early 1980s, and effective numerical methods have been developed for practical implementation in engineering applications. A major remaining issue is to develop a corresponding theory for nonlinear control systems. This book presents a systematic account of recent progress in this regard by the authors and others.

Linear H_∞ control theory can be considered in either a frequency domain, input-output formulation or a time domain, state space formulation. Mathematical tools of the linear theory involve such techniques from operator theory and complex function theory as Nevanlinna-Pick interpolation and inner-outer factorizations. Matrix Riccati equations also have a key role. In contrast, nonlinear H_∞ control theory is formulated in the time domain and depends on ideas and methods of differential games and nonlinear partial differential equations (or partial differential inequalities).

The book considers a standard nonlinear H_∞ control formulation, as follows. Let x_t denote the system state, u_t a control, and w_t a disturbance at time $t \geq 0$. The state dynamics are

$$\dot{x}_t = A(x_t) + B_1(x_t)w_t + B_2(x_t)u_t, \tag{1}$$

with $A(x)$, $B_1(x)$, $B_2(x)$ matrices of appropriate dimensions. The disturbance w_t is not known by the controller. The control u_t is chosen based on available information. Two cases are considered, called state-feedback and output-feedback control. The control must be chosen such that the undisturbed system, with $w_t \equiv 0$, is asymptotically stable to 0. Moreover, for suitable $\gamma > 0$ the disturbed system must be γ-dissipative in the following sense. Consider a performance measure z_t, with

$$z_t = C_1(x_t) + D_1(x_t)u_t \tag{2}$$

where $D_1(x)D_1(x)' > 0$. It is required that there exists $\beta(x) \geq 0$ with $\beta(0) = 0$ such that: for every $T > 0$, x_0 and $w_. \in L_2[0, T]$

$$\frac{1}{2}\int_0^T |z_s|^2 ds \leq \frac{1}{2}\gamma^2 \int_0^T |w_s|^2 ds + \beta(x_0). \tag{3}$$

The infimum of such γ is the H_∞ norm.

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<u>State-feedback</u> $\mathbf{H}_\infty$ <u>control</u>.

Assume that the state x_t can be observed by the controller, and take $u_t = K(x_t)$ where $K(\cdot)$ is a state-feedback controller. In linear state-feedback control theory the matrices $A, B_1, \cdots$ in equations (1) and (2) are constants, and $K(x)$ is linear. If $x_0 = 0$, then $\beta(x_0) = 0$ and inequality (3) says that γ is an upper bound for the $L_2[0,T]$ operator norm of the linear map $w_\cdot \to z_\cdot$, for each T. In passing to the frequency domain, the operator norm becomes a H_∞ norm. The name "H_∞ control" has persisted in the nonlinear setting even though this frequency domain interpretation is no longer valid.

Given a feedback controller K and initial state x_0, let

$$J_T = \tfrac{1}{2}\int_0^T \left[|z_s|^2 - \gamma^2|w_s|^2\right] ds$$

$$V_K(x_0) = \sup_T \sup_{w_\cdot} J_T.$$

Finiteness of $V_K(x_0)$ for all x_0 implies the dissipation inequality (3) with $\beta \geq V_K$. The function V_K is called available storage. It satisfies, in the viscosity sense, a first order nonlinear PDE of Hamilton-Jacobi-Bellman (HJB) type. There is a more easily verified sufficient condition for (3) which requires some storage function (not necessarily the available storage V_K) satisfying in the viscosity sense a corresponding partial differential inequality.

If γ exceeds the H_∞ norm, one should expect many state feedback controllers K with the required stability and γ-dissipation properties. The following provides a recipe for one such K, called the central controller. Let

$$V(x_0) = \inf_K V_K(x_0).$$

Under suitable technical assumptions $V(x_0)$ is the value of a corresponding zero-sum two player differential game, in which the minimizing player chooses u_t and the maximizing player chooses the disturbance w_t. Moreover, V is a viscosity solution of the Isaacs PDE. Under suitable regularity assumptions, the central controller $K^*(x)$ is obtained by taking argmax over possible controls u in the Isaacs equation.

<u>Output-feedback</u> $\mathbf{H}_\infty$ <u>control</u>.

The main part of the book is concerned with the situation when the state x_t cannot be observed. Instead, at time t the control u_t is chosen based on observations y_s for $0 \leq s \leq t$, where

$$y_s = C_2(x_s) + D_2(x_s)w_s\ , \qquad D_2D_2' > 0. \tag{4}$$

The controller K is now a casual map $y_\cdot \to u$. An essential feature of the authors' approach is to recast the output-feedback problem in a state space setting, which can then be analyzed by PDE/differential game methods which are formally similar to those for the state feedback case. However, in the reformulated problem the "state" is not x_t, which can't be observed. Instead it is an "information state" function p_t, defined as follows:

$$p_t(x) = \sup_{w_\cdot} \{p_0(x_0) + J_t \,:\, x_t = x\} \tag{5}$$

where the sup is taken over all disturbances consistent with (4), and p_0 is an initial information state with $p_0(x) \leq 0$, $p_0(0) = 0$. Intuitively, the information state function contains all past information relevant to future evolution of the output-feedback control system. The information state evolves forward in time according to an HJB partial differential equation, interpreted in the viscosity sense.

A key observation is that the γ-dissipation property (3) is equivalent to $\langle p_T \rangle \le 0$ for all T and $y_\cdot \in L_2[0,T]$, where

$$\langle p_T \rangle = \sup_x p_T(x).$$

An information state value function $W(p_0)$ is defined as

$$W(p_0) = \inf_K \sup_{T,y_\cdot} \langle p_T \rangle,$$

where K is an information state feedback controller ($u_t = K(p_t)$). Formally, $W(p)$ satisfies an infinite dimensional nonlinear PDE. However, this PDE does not fit well existing infinite dimensional viscosity solution theory. One can interpret it as the Isaacs PDE for a differential game, with p_t as state and u_t, y_t controls chosen by the minimizing and maximizing players respectively. Under enough technical assumptions a central information state controller $K^*(p)$ is found in a way formally like that for the state feedback central controller $K^*(x)$. It has the property that $K^*(p_e) = 0$ where p_e is an equilibrium information state, which remains constant in time if $u_t = y_t \equiv 0$. More detailed information about p_e is found in particular cases. For instance, if $D_2(x)$ is the identity matrix (a "two block system" case) and if $A - B_1 C_2$ is a hyperbolic vector field, then $p_e(x)$ is δ-function like on its unstable manifold and vanishes elsewhere.

Another technique is that of inner-outer factorization, in which the inner factor is dissipative and the outer factor satisfies a weak invertibility condition. A recipe for the factors, in the information state context, is provided.

For bilinear systems, the information states $p_t(x)$ are quadratic in x and the information-state level analysis becomes finite dimensional. In particular, for linear systems the analysis reduces to questions about matrix Riccati equations.

Another way to avoid the infinite dimensional PDE framework is to consider "certainty-equivalent" controllers $u_t = K(\hat{x}_t)$, where $\hat{x}_t$ is an estimate for the unknown state x_t and K is a state-feedback controller. The certainty equivalent controller chooses $K(x) = K^*(x)$, an optimal state-feedback controller, and $\hat{x}_t = \bar{x}(p_t)$, where

$$\bar{x}(p) = \underset{x}{\operatorname{argmax}} \{p(x) + V(x)\} \tag{6}$$

and $V(x)$ is the state-feedback value function. This corresponds to considering the solution $\hat{W}(p) = \langle p + V \rangle$ of the infinite dimensional HJB equation for $W(p)$. Under suitable assumptions, including uniqueness of the argmax in (6), the certainty equivalent and central information state controllers agree.

This book provides a definitive account of the information state approach to nonlinear H_∞ control. The introductory Chapter 1 gives a valuable, less technical overview as well as historical perspective. The book is also a good resource for an entree to basic concepts and techniques of nonlinear H_∞ control, in both state-feedback and output-feedback settings.

References

1. M. Bardi and I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Birkhauser, Boston, 1997. MR **99e:**49001
2. T. Basar and P. Bernhard, H_∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach, Birkhauser, Boston, 1995 (2nd Ed.) MR **96f:**93002
3. J. Doyle, B.A. Francis and A. Tannenbaum, Feedback Control Theory, Macmillan Publ. Co., New York, 1992. MR **93k:**93002
4. M. Green and D.J.N. Limebeer, Linear Robust Control, Prentice-Hall, Englewood Cliffs, N.J., 1995.
5. J.W. Helton, J. Ball, C. Johnson and C. Palmer, Operator Theory, Analytic Functions, Matrices and Electrical Enginering, Regional Conference Series in Math. vol 68, Amer. Math. Soc., 1987. MR 89f:47001
6. A.J. van der Schaft, L_2-gain and Passivity Techniques in Nonlinear Control, Springer-Verlag, New York, 1996. MR **97e:**93002
7. P. Whittle, Optimal Control: Basics and Beyond, John Wiley and Sons, Chichester, England, 1996. MR **98b:**49001

Wendell H. Fleming
Brown University
E-mail address: whf@cfm.brown.edu