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On the local structure of Morita and Rickard equivalences between Brauer blocks, by Lluis Puig, Progress in Math., vol. 178, Birkhäuser, Basel, 1999, 261 pp., $98.00, ISBN 3-7643-6156-5

This book is concerned with block theory, which is one of the main subjects of the modular representation theory of finite groups. There are several notions of equivalences between blocks, so in order to give a flavour of the subject, we shall first explain what is a Morita equivalence, a Rickard equivalence, a stable equivalence, and a Puig equivalence. The main results of the book show how far each equivalence is from the last type, which is the strongest of them.

Although the interplay between representations in characteristic zero and representations in characteristic p is crucial, let us consider for simplicity only representations over a fixed field K, which we also assume to be algebraically closed. The group algebra of a finite group G with coefficients in K is the K-algebra KG having G as a basis, with bilinear multiplication induced by the product of basis elements. A KG-module is also called a *representation* of G over K. We assume that all modules are finitely generated, and this amounts here to the condition that they have finite dimension as K-vector spaces.

In the classical theory, the characteristic of K is zero, and the group algebra KG turns out to be isomorphic to a direct product of matrix algebras $KG \cong \prod_{i=1}^{r} M_{n_i}(K)$. Moreover, any KG-module V can be written $V = \bigoplus_{i=1}^{r} V_i$, where V_i is a module over $M_{n_i}(K)$ (with zero action of the other factors of the product). Now there is only one simple $M_{n_i}(K)$-module S_i up to isomorphism, and every $M_{n_i}(K)$-module is isomorphic to a direct sum of copies of S_i. This reduces the classification of KG-modules to the listing of the r distinct simple modules S_i, called the *irreducible* representations of G over K. In other words, the category mod (KG) of all KG-modules decomposes as the direct product of the categories mod $(M_{n_i}(K))$, and each of them is quite easy to understand. In fact mod $(M_{n_i}(K))$ is equivalent to the category of K-vector spaces.

Assume from now on that the characteristic of K is a prime number p. We cannot always decompose KG as a direct product of matrix algebras, but we can obviously decompose it as much as possible. We let $KG \cong \prod_{j=1}^{m} B_j$ be the finest possible decomposition as a direct product (which is unique up to isomorphism), and we call B_j a *block algebra*, or simply a *block* of KG. One of the main goals of modular representation theory is to understand the structure of a block algebra B and of the associated module category mod (B). By the Krull–Schmidt theorem, every module decomposes as a direct sum of indecomposable summands in a unique way up to isomorphism, but it should be noted that there are in general infinitely many non-isomorphic indecomposable B-modules. Thus the module category of B can be considerably more complicated than that of a matrix algebra, which was the only case occurring in characteristic zero.

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Now we can explain one crucial idea of block theory. When one allows the group G to vary (for instance in some specific class of finite groups), there are numerous examples of an infinite family of blocks which all look the same: they may have equivalent module categories, or they may have identical behaviour as far as character values are concerned. So they are in some sense equivalent, but several notions of equivalence actually occur.

Two algebras A and A' are called *Morita equivalent* if their module categories mod (A) and mod (A') are equivalent. By a classical theorem of Morita, such an equivalence can always be obtained by tensoring with an (A', A)-bimodule. The notion of derived equivalence is defined by replacing mod (A) by a suitable category of *complexes* of A-modules, called the *derived category* of A. Two algebras A and A' are called *derived equivalent* if their derived categories are equivalent. In analogy with the Morita theorem, there is a theorem of Rickard [Ri1] which asserts that a derived equivalence can always be obtained by tensoring with a complex of (A', A)-bimodules. For this reason, a derived equivalence is now often called a *Rickard equivalence*. Finally A and A' are called *stably equivalent* if their stable categories are equivalent, where the stable category is defined to be the quotient of the module category obtained by killing all projective modules. Since there is no theorem available analogous to the Morita or Rickard theorems, we need the stronger notion of *Morita stable equivalence*, which is defined by requiring that the stable equivalence is induced by a bimodule. Each of our three types of equivalences implies the next; that is, if A and A' are Morita equivalent, then they are Rickard equivalent, and if they are Rickard equivalent, then they are Morita stably equivalent.

There are important examples of block algebras where there is an equivalence which resembles a Morita equivalence except that some signs occur (i.e. a module is sent to a difference of modules) and it turns out that the equivalence is in fact a Rickard equivalence: the difference of modules has to be interpreted as an alternating sum of modules appearing in a complex. This explains briefly why Rickard equivalences have forced their way into block theory (see [B1], [B2], [Ri2], [KZ]). Actually, one of the interesting open problems in this subject is a conjecture of Broué, which predicts the existence of a certain Rickard equivalence for blocks with an abelian defect group (see [B1], [B2]). Although this does not occur explicitly in Puig's book, we mention that the notion which seems to play a particularly important role in block theory is that of *splendid equivalence* (see [Ri2]), which is a Rickard equivalence induced by a special type of complex of bimodules, so that the equivalence has specific properties with respect to the p-subgroups of the group (there is a similar equivalence induced at the p-local level, that is, between corresponding blocks of centralizers of p-subgroups).

While our main three definitions make sense for all algebras, the next notion of equivalence, which is the strongest, only has a meaning for block algebras, for it involves the group structure. In order to define it, we first need to introduce source algebras. A first invariant which measures the complexity of a block of KG is its *defect group*, introduced by Richard Brauer in the fifties. It is a p-subgroup of G (unique up to conjugation), hence sandwiched somewhere between the trivial subgroup and a Sylow p-subgroup. The next invariant associated with a block algebra is its *source algebra*, which is unique up to isomorphism. It is a finite dimensional algebra in which the defect group P is embedded (so in particular P acts on the source algebra by conjugation). This invariant was introduced by

Lluis Puig 20 years ago [P1] (see also the textbook [Th]). The source algebra is a subalgebra of the group algebra KG (with a different identity element), but the main idea is that G has disappeared and only p-local information (i.e. information concerned with p-subgroups) remains in the source algebra. A block algebra is Morita equivalent to its source algebra, and moreover all known p-local invariants of a block can be detected from its source algebra. So the source algebra faithfully reflects the structure of the block.

Two blocks are called *Puig equivalent* if they have the same defect group and isomorphic source algebras. This definition immediately leads to the question of classifying blocks up to Puig equivalence, that is, classifying all possible source algebras for a given defect group P. It is conjectured by Puig that, for a given p-group P, there are only finitely many possible source algebras of blocks, hence finitely many Puig equivalence classes of blocks with defect group P. However, the classification problem is a hard one which is far from being solved. Many properties of source algebras of blocks are known, but they do not suffice yet to characterize them (see [Th] for an introduction).

For example, the most elementary possibility occurs if the block algebra B happens to be isomorphic to a matrix algebra $M_n(K)$ (in which case there is a unique indecomposable B-module, which is also simple). In that case, the defect group is trivial and the source algebra is just the one-dimensional algebra K. Any Chevalley group in natural characteristic p always has a block of this type, whose unique simple module is the Steinberg module. Also, if p does not divide the order of the group G, each block is of this form, and we are in the same situation as in characteristic zero. So we really only have to deal with groups of order divisible by p.

Let us mention a few results about source algebras of blocks. For blocks with a cyclic defect group, Puig's conjecture has been recently proved by M. Linckelmann [L1], using structural theorems proved by E.C. Dade more than 30 years ago. The structure of source algebras has also been described when the defect group is a Klein four group [L2], when the block is nilpotent [P2] (see also [Th]), when the group is p-solvable, and for some blocks of Chevalley groups [P3]. Weaker forms of Puig's conjecture have also been proved, for instance for blocks of p-solvable groups only, or symmetric groups only [P4].

For block algebras, there are now at least 4 different kinds of equivalences: the Puig equivalence, the Morita equivalence, the Rickard equivalence, and the Morita stable equivalence. Each equivalence implies the next. There are important examples of pairs of blocks for which it is known that only the second, or the third, or the fourth equivalence holds. So one may ask how far these cases are from the first one, which is the most demanding.

In the book under review, L. Puig answers this question by showing how to measure the distance between a Morita equivalence (or a Rickard equivalence, or a Morita stable equivalence) and a Puig equivalence. Slightly more precisely, it is shown how to obtain the source algebra of one of the blocks from the source algebra of an equivalent block and invariants coming from the equivalence itself (namely from the bimodule or the complex of bimodules which induces the equivalence). In the case of a Rickard equivalence, this requires one to consider DG-modules rather than KG-modules, where D is an algebra which encodes both the grading of complexes and the differential between the successive terms of the grading.

The precise statements of the results are too technical to be given here explicitly, and of course they depend on the kind of equivalence which comes into play. For each of the main results, the idea is to embed one of the source algebras into an algebra constructed from the other source algebra. Among other things, the construction involves a general notion of induction of algebras, which works for an arbitrary group homomorphism $H \to G$ rather than an inclusion $H \subset G$. In the case of a Morita or stable Morita equivalence, the construction also involves a tensor product with the endomorphism algebra of a suitable module N (a source of the bimodule which induces the equivalence). Actually, the module N turns out to be an endo-permutation module in a special case of equivalence, called a *basic* equivalence, which is apparently the most common case. In the case of a Rickard equivalence, all this also applies, but another construction replaces the module N.

As a by-product of the results, it is shown that two blocks are Puig equivalent if and only if they are Morita equivalent via a bimodule which is a direct summand of a permutation module (a so-called trivial source module), and also that a block which is Morita stably equivalent to a nilpotent block must again be nilpotent.

The style of the book is that of a long research paper. It is not an exposition of the subject nor a report on recent work in this area. It is based on the author's previous papers, and so the reading requires a good acquaintance with Puig's work and also some ideas about other people's work, at least for motivation. General ideas, background results, and motivation about the subject of the book, in particular about the methods, are rather well explained in the introduction, where one can also find the connection with some of the recent work of other mathematicians.

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