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Frobenius manifolds, quantum cohomology, and moduli spaces, by Yuri I. Manin, Amer. Math. Soc., Providence, RI, 1999, xiii + 303 pp., $55.00, ISBN 0-8218-1917-8

Geometric structures on spaces X are often reflected in algebraic structures on the cohomology $H^*(X)$. For example, if X is a compact oriented manifold, $H^*(X)$ has a non-degenerate graded symmetric bilinear form, while if X is a quasi-projective variety over $\mathbb{C}$, $H^*(X)$ has a mixed Hodge structure.

What algebraic structures exist on $H^*(X)$ when X is a compact symplectic manifold and, in particular, a smooth projective variety? Inspired by Gromov's theory of pseudoholomorphic curves [10], Witten's work on topological gravity [15] and an influential paper of Ruan [13], the past few years have seen the construction of a rather complicated algebraic structure on the cohomology of such a manifold, the Gromov-Witten invariants of X.[1]

Formally, this structure has some resemblance to the action of Steenrod operations on $H^*(X, \mathbb{F}_p)$, except that these are linear, while the operations of Gromov-Witten theory are multilinear. Frobenius manifolds were introduced by Dubrovin in order to provide a "coordinate free" approach to this structure. Manin's book is a beautiful survey of the theory of Gromov-Witten invariants, using the framework of Frobenius manifolds and of the theory of Frobenius manifolds itself (much of it based on his research and that of his collaborators).

Frobenius manifolds actually arise in a number of different areas, such as singularity theory, integrable systems, and homological algebra, and much insight is gained by considering them from all of these points of view. An especially attractive feature of Manin's book is the large number of examples of Frobenius manifolds drawn from all of these areas of algebra and geometry, as well as background essays on isomonodromy (Chapter II), operads and their generalizations (Chapter IV) and intersection theory on stacks (Chapter V), rendering the book accessible to a wider audience.

In this review, we can do little more than touch on some of the examples and properties of Frobenius manifolds and Gromov-Witten invariants; a far better introduction to the subject is contained in Manin's introductory Chapters 0 and I, on Gromov-Witten theory and Frobenius manifolds respectively.

The recent book by Cox and Katz [2] focuses on explicit calculations of Gromov-Witten invariants of smooth projective varieties and complements Manin's book well. For other accounts of topics in the theory of Frobenius manifolds and Gromov-Witten invariants, see Behrend [1], Dubrovin [3], [4], Fulton and Pandharipande [8] and Hitchin [12].

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[1] This is what Manin calls quantum cohomology; because of ambiguities in the literature over the meaning of this term, we prefer to avoid it. In any case, the word "quantum" is probably overused nowadays.

Frobenius manifolds

Let M be a complex manifold. (In applications, it is often a "germ of a complex manifold" or a "formal complex manifold".) In any of these settings, there is an analogue of Riemannian geometry: a holomorphic metric is a non-degenerate holomorphic symmetric bilinear form on the complex tangent bundle. (The condition of definiteness on the metric is not meaningful in the complex domain.)

To such a metric is associated a Levi-Civita connection (which is not a connection in the usual sense, but rather a holomorphic connection) and its curvature. We say that a holomorphic metric is flat if its curvature vanishes; this is equivalent to the local existence of complex coordinates t^a in which the metric becomes constant.

A Frobenius manifold, as defined by Dubrovin [3], is a complex manifold M with the following additional data:

1) a flat holomorphic metric $g \in \Gamma(\mathrm{Sym}^2 T^*M)$, with Levi-Civita connection ∇;
2) a tensor $A \in \Gamma(\mathrm{Sym}^3 T^*M)$;
3) a flat vector field $e \in \Gamma(TM)$.

These are subject to the following conditions:

a) the product $X \circ Y$ on TM, defined by $g(X \circ Y, Z) = A(X, Y, Z)$, makes TM into a bundle of commutative associative algebras, with e as identity;
b) locally, there is a potential function Φ such that $A(X, Y, Z) = X(Y(Z(\Phi)))$ for flat vector fields X, Y and Z.

The associativity of the product $X \circ Y$ is a differential equation for Φ, called the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation. Note that g is determined by A and e, by the equation $g(X, Y) = A(e, X, Y)$, so that the structure is actually determined by Φ and e.

An important discovery of Dubrovin is that the axioms of a Frobenius manifold (other than those relating to the identity vector field e) may be reformulated as the flatness of a pencil of connections

$$\nabla^\lambda_X Y = \nabla_X Y + \lambda\, X \circ Y. \tag{1}$$

This connection plays an important role in the classification of Frobenius manifolds and in the study of their geometry.

The definition of a Frobenius manifold is a little difficult to digest: the best approach is perhaps to study some examples.

Constant Frobenius manifolds. Let V be a unital commutative algebra with a non-degenerate symmetric bilinear form η, such that $\eta(ab, c) = \eta(a, bc)$. (In other words, V is a commutative Frobenius algebra.) Then V is a Frobenius manifold, with cubic potential function $\Phi(a) = \frac{1}{6}\eta(a, a^2)$. This example motivated Dubrovin's choice of terminology.[2]

The examples of Frobenius manifolds which arise in Gromov-Witten theory are deformations of Frobenius manifolds of this type, where the commutative algebra is $H^*(X)$ and the inner product is the Poincaré form. (Of course, if $H^*(X)$ has odd-dimensional elements, we must speak of Frobenius supermanifolds.)

[2]On the other hand, since Frobenius algebras need not be commutative, perhaps Frobenius manifolds should have been named Saito manifolds — but it is probably too late to modify the established terminology.

Two-dimensional Frobenius manifolds. Let M be a two-dimensional Frobenius manifold such that $g(e,e) = 0$. We may choose coordinates s,t on M such that $e = \partial_s$ and $g = ds\,dt$; the potential Φ equals

$$\Phi = \tfrac{1}{2}\,s^2\,t + \phi(t).$$

The axioms for a Frobenius manifold place no restriction on $\phi(t)$.

Three-dimensional Frobenius manifolds. Let M be a three-dimensional manifold with coordinates $\{s,t,u\}$, vector field $e = \partial_s$, metric $g = ds\,du + \frac{1}{2}\,dt^2$, and potential function

$$\Phi = \tfrac{1}{2}\,(s^2 u + s\,t^2) + \phi(t,u).$$

The existence of a potential function is equivalent to the equation

$$\phi_{uuu} = (\phi_{ttu})^2 - \phi_{ttt}\phi_{tuu}.$$

K. Saito's flat structures. Let A_n be the space of monic polynomials whose derivative has non-zero discriminant

$$A_n = \{f(z) = z^{n+1} + \textstyle\sum_{i=1}^n a_i z^{n-i} \mid \Delta(f') \neq 0\}.$$

The tangent space $T_f A_n$ at a polynomial $f \in A_n$ may be identified with $\mathbb{C}[z]/(f')$; in this way, the tangent sheaf of A_n becomes a sheaf of commutative algebras, with identity $e = \partial/\partial a_n$.

The manifold A_n carries a holomorphic metric

$$(a,b)_f = -\operatorname{Res}_\infty(a\,b/f') = \sum_{f'(\rho)=0} \frac{a(\rho)b(\rho)}{f''(\rho)}.$$

Saito shows (in a more general setting [14]) that this holomorphic metric on A_n is flat and that all of the axioms of a Frobenius manifold hold. These were the first examples of Frobenius manifolds.

Gromov-Witten invariants

The deepest construction of Frobenius manifolds, which provides the *raison d'être* for Manin's book, comes from Gromov-Witten theory, whose outlines we recall.

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne-Mumford-Knudsen moduli space of stable curves of genus g with n marked points, where $2g-2+n > 0$. This is a connected compact complex orbifold of complex dimension $3g-3+n$. The most familiar of these moduli spaces are $\overline{\mathcal{M}}_{0,4}$, which by the cross-ratio is isomorphic to the sphere, and $\overline{\mathcal{M}}_{1,1}$, which is the compactified moduli space of elliptic curves. A good place to learn about these spaces is Harris and Morrison [11].

Let (X,ω) be a compact symplectic manifold of dimension $2d$, with first Chern class $c_1(X)$. The Novikov ring Λ of X is a graded topological ring consisting of those series

$$f = \sum_{\beta \in H_2(X,\mathbb{Z})} a_\beta\, q^\beta \in \mathbb{C}[[H_2(X,\mathbb{Z})]]$$

such that for each $A > 0$, the set $\operatorname{supp}(f) \cap \{\beta \in H_2(X,\mathbb{Z}) \mid \omega(\beta) < A\}$ is finite; the grading is determined by $\deg(q^\beta) = 2\,c_1(X)(\beta)$. For example, if $X = \mathbb{CP}^d$, $\Lambda = \mathbb{C}[\![q]\!]$, where $\deg(q) = 2(d{+}1)$. The cohomology $H^*(X,\Lambda)$ of X with coefficients in Λ is known as the Floer homology of X.

The Gromov-Witten invariants are multilinear maps

$$I^X_{g,n} = \sum_{\beta \in H_2(X,\mathbb{Z})} q^\beta \, I^X_{g,n,\beta} : H^*(X,\Lambda)^{\otimes n} \longrightarrow H^{*-2\,d(1-g)}(\overline{\mathcal{M}}_{g,n},\Lambda),$$

defined by enumerating (in a suitably generalized sense) the pseudoholomorphic curves in X of genus g and degree β, with homological constraints imposed at n points of the curve. (Note that the sum over β may be infinite, but it converges in the topology of Λ.) Their most important property is that they are independent of the almost-complex structure used in their construction.

The construction of Gromov-Witten invariants is based on Gromov's idea that it is not curves $C \subset X$ but rather maps $f : C \to X$ which are to be studied; this simple idea has had a truly revolutionary effect in algebraic geometry. Kontsevich observed that compactifications arising in Gromov's theory of pseudoholomorphic maps are similar to the Deligne-Mumford-Knudsen moduli spaces. Motivated by this, he introduced for a smooth projective variety X the moduli stacks $\overline{\mathcal{M}}_{g,n}(X)$ of stable maps from n-pointed projective curves of arithmetic genus g to X.

For generalized flag varieties, the genus 0 Gromov-Witten invariants are now easy to construct: $\overline{\mathcal{M}}_{0,n}(X)$ is a smooth stack, and the Gromov-Witten invariant is represented by its fundamental class.

The construction of Gromov-Witten invariants in general is very lengthy and is the culmination of work by many people. Manin presents a construction of Gromov-Witten invariants for smooth projective varieties due to Behrend and Fantechi. (The subject is unavoidably technical, and he does an excellent job of introducing the requisite stack theory in as painless a way as possible.) Manin does not touch on the construction of Gromov-Witten invariants for compact symplectic manifolds and the related subject of Floer homology, but the framework which he presents for Gromov-Witten theory applies more or less verbatim in the symplectic case, so his book should find a readership among symplectic geometers too.

Gromov-Witten invariants in genus 0 and formal Frobenius manifolds. Let M be the formal manifold (defined over the Novikov ring Λ) obtained by completing the affine space $H^*(X,\Lambda)$ at the origin. (If $H^*(X,\mathbb{C})$ has elements of odd degree, this is actually a supermanifold.)

Let $\{\gamma_a \in H^{d_a}(X,\mathbb{C})\}$ be a homogeneous basis of $H^*(X,\mathbb{C})$ with dual basis t^a; the coordinate ring of M is $\Lambda[\![t^a]\!]$. Assume that γ_0 is the identity $1 \in H^0(X,\mathbb{C})$. It follows easily from the properties of the genus 0 Gromov-Witten invariants that the following data define the structure of a Frobenius manifold on M:

a) g is the flat holomorphic metric $g = \frac{1}{2}\sum_{a,b} \eta_{ab} dt^a\, dt^b$ on M, where $\eta_{ab} = \int_M \gamma_a \cup \gamma_b$ are the components of the intersection form of X;
b) the potential Φ is

$$\Phi = \sum_{n=3}^{\infty} \frac{1}{n!} \sum_{a_1,\dots,a_n} t^{a_1} \dots t^{a_n} \, I^X_{g,n}(\gamma_{a_1},\dots,\gamma_{a_n});$$

c) the identity vector field is $e = \partial/\partial t^0$.

Less is known about the structure of Gromov-Witten invariants in genus $g > 0$ than in genus 0. Intriguingly, Dubrovin and Zhang [6] have shown that the relations which hold among genus 1 Gromov-Witten invariants take a very simple form when formulated on a semisimple Frobenius manifold. It is possible that this is the case in higher genus as well; for further references, see Dubrovin [5].

A problem in enumerative geometry. Of course, Gromov-Witten invariants have many applications in enumerative geometry, such as the enumeration of rational curves in projective spaces. The resulting calculation is especially illuminating when formulated in the language of Frobenius manifolds.

Curves of genus g and degree d in the projective plane $\mathbb{CP}^2$ are parametrized by a variety of dimension $3d+g-1$. The subvariety of curves passing through $3d+g-1$ general points is a reduced variety of dimension 0; its cardinality $N_d^{(g)}$ is an invariant, and the calculation of these numbers is a classic problem of algebraic geometry. For example, $N_1^{(0)}$ is the number of lines in the plane passing through 2 general points, which is of course equal to 1, while $N_2^{(0)}$ is the number of conics passing through 5 general points, also equal to 1.

These numbers may be expressed using Gromov-Witten invariants:

$$N_d^{(g)} = \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n,d}^{\mathbb{CP}^2}\big(\Omega^{\otimes(3d+g-1)}\big),$$

where $\Omega \in H^4(\mathbb{CP}^2, \mathbb{Z})$ is the cohomology class Poincaré dual to a point. In particular, the genus 0 numbers $N_d = N_g^{(0)}$ determine the potential function $\Phi = \frac{1}{2}(s^2u + st^2) + \phi(t,u)$ of the Frobenius manifold associated to the Gromov-Witten invariants of $\mathbb{CP}^2$: we have

$$\phi(t,u) = \sum_{d=1}^{\infty} N_d \frac{q^d e^{td} u^{3d-1}}{(3d-1)!}.$$

The WDVV equation $\phi_{uuu} = (\phi_{ttu})^2 - \phi_{ttu}\phi_{tuu}$ amounts to a recursion

$$N_d = \sum_{e=1}^{d-1} \Big(\tbinom{3d-4}{3e-2} e^2 (d-e)^2 - \tbinom{3d-4}{3e-1} e^3 (d-e) \Big) N_e N_{d-e}$$

which uniquely determines the series N_d in terms of $N_1 = 1$. The same method calculates the rational Gromov-Witten invariants of all generalized flag varieties spaces and, in particular, of projective spaces of any dimension.

Frobenius manifolds as algebras

Formal Frobenius manifolds may be thought of as algebraic objects of the following type. Let A be a vector space with nondegenerate symmetric bilinear form η, a sequence of symmetric products $(x_1, \dots, x_n) : A^{\otimes n} \to A$, $n \geq 2$, and a vector $e \in A$. Then A is a formal Frobenius manifold, with metric η, potential function

$$\Phi(t) = \sum_{n=3}^{\infty} \frac{1}{n!}\, \eta(t, (\underbrace{t, \dots, t}_{n-1 \text{ times}})),$$

and identity vector e, provided the following conditions are met.

a) The products satisfy a generalized associativity condition: for elements a, b, c, x_1, $\dots$, $x_n \in A$,

$$\sum_{I \amalg J = \{1,\dots,n\}} ((a,b,x_I), c, x_J) = \sum_{I \amalg J = \{1,\dots,n\}} (a, (b,c,x_I), x_J).$$

Here, if $I = \{i_1, \dots, i_k\}$ is a finite set, x_I is an abbreviation for $x_{i_1}, \dots, x_{i_k}$.

b) The products are compatible with the inner product η, in the sense that $\eta(a_0, (a_1, \dots, a_n)) = \eta((a_0, \dots, a_{n-1}), a_n)$.
c) The vector e is an identity, in the sense that $(e, a) = a$ and $(e, a_1, \dots, a_n) = 0$ for $n > 1$.

In particular, the product (a, b) is a commutative, associative product with identity e, while

$$((a, b, x), c) + ((a, b), c, x) = (a, (b, c, x)) + (a, (b, c), x).$$

This point of view on formal Frobenius manifolds motivates the definition analogues of a number of the constructions of commutative algebra. In particular, there is an analogue for Frobenius manifolds of Harrison cohomology and its cyclic analogue, giving rise to a deformation theory of Frobenius manifolds. The study of this deformation complex in turn motivated a construction of Frobenius manifolds from a Batalin-Vilkovisky algebra (due to Barannikov and Kontsevich and explained in Chapter III of Manin's book), which is a promising approach to generalizing mirror symmetry to higher dimensions.

Another analogy with commutative algebra is the tensor product of formal Frobenius manifolds, defined by Kaufmann in terms of the diagonal $H_*(\overline{\mathcal{M}}_{0,n}) \mapsto H_*(\overline{\mathcal{M}}_{0,n}) \otimes H_*(\overline{\mathcal{M}}_{0,n})$. The tensor product of the Frobenius manifolds associated to the Gromov-Witten invariants of manifolds V and W is the Frobenius manifold associated to the Gromov-Witten invariants of $V \times W$; this is the Künneth theorem for Gromov-Witten invariants.

In a remarkable example of the interplay between the algebraic and analytic points of view on Frobenius manifolds, the tensor product of two analytic germs of Frobenius manifolds is again an analytic germ of a Frobenius manifold. For example, $A_2 \otimes A_3$ is the Frobenius manifold associated to the singularity E_6.

Euler vector fields

In all constructions of Frobenius manifolds, there is a further piece of structure, called an Euler vector field, which plays a role similar to the grading in a graded commutative algebra. The structure generated by this vector field has proved to be essential in the study of Frobenius manifolds.

An Euler vector field E on a Frobenius manifold M is a vector field such that $\mathcal{L}_E A = (3-d)A$, where d is a real number, and $[E, e] = -e$. Equivalently, an Euler vector field satisfies the formulas

$$E(g(X, Y)) = g([E, X], Y) + g(X, [E, Y]) + (2-d)\, g(X, Y), \quad \text{and}$$
$$[E, X \circ Y] = [E, X] \circ Y + X \circ [E, Y] + X \circ Y.$$

Euler vector fields for two-dimensional Frobenius manifolds. As an illustration, let us classify the possible Euler vector fields on a two-dimensional Frobenius manifold. The formulas $\mathcal{L}_E g = (2-d)g$ and $[E, e] = -e$ imply that

$$E = (s + a)\partial_s + ((1-d)t + b)\partial_t.$$

There are two cases to consider:

i) $d \neq 1$: We may redefine s and t so that $E = s\,\partial_s + (1-d)t\,\partial_t$. The equation $\mathcal{L}_E A = (3-d)A$ implies that $A = \frac{1}{2}\, ds^2\, dt + c\, t^{2d/(1-d)}\, dt^3$.

ii) $d = 1$: We may redefine s and t so that $E = s\,\partial_s + 2\,\partial_t$; it follows that $\phi(t) = e^t$. This example is related to the Frobenius manifold associated to the genus 0 Gromov-Witten invariants of $\mathbb{CP}^1$.

Euler vector fields for Gromov-Witten theory. Using the fact that the Gromov-Witten invariant $I^X_{g,n}$ of a symplectic manifold X has degree $-2\,d(1-g)$, one can show that the Euler vector field for the Frobenius manifold associated to the genus 0 Gromov-Witten invariants of X is given by the formula

$$E = \sum_a \big((1-d_a)t^a + R^a\big)\frac{\partial}{\partial t^a}, \quad \text{where } R^a = \textstyle\sum_b \eta^{ab} \int c_1(V) \cup \gamma_b.$$

In this case, the constant d may be identified with the complex dimension of X.

Semisimple Frobenius manifolds. A Frobenius manifold is semisimple if the endomorphism $X \mapsto E \circ X$ of the tangent bundle is semisimple and the set of points $U \subset M$ at which its eigenvalues $\{u^i\}$ form a coordinate system is dense.[3] These coordinates are called the canonical coordinates, and in terms of them, the product and metric take a remarkably simple form: $\partial/\partial u^i \circ \partial/\partial u^j = \delta_{ij}\,\partial/\partial u^i$, and

$$g = \tfrac{1}{2} \sum_i \frac{\partial \eta}{\partial u^i}\,(du^i)^2, \quad \text{where } \eta = \textstyle\sum_a g_{0a} t^a.$$

In canonical coordinates, the Frobenius manifold is determined by the function η, and the differential equations which it must satisfy in order for the metric g to be flat form an integrable system. By analysing this integrable system, Dubrovin has classified semisimple Frobenius manifolds. Manin discusses this classification, and its relationship with the Painlevé VI equation in the case of a three-dimensional Frobenius manifold.

A Virasoro algebra. By a theorem of Hertling and Manin, an Euler vector field on a Frobenius manifold M gives rise to an action of the Lie algebra of vector fields on the line on M:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n},$$

where $\mathcal{L}_{-1} = -e$ and $\mathcal{L}_{n+1} = E \circ \mathcal{L}_n$. The Virasoro conjecture of Eguchi, Hori and Xiong [7] hints that this Lie algebra also plays a central role in the axiomatization of higher genus Gromov-Witten invariants.

In canonical coordinates, the identity vector field e and Euler vector field E equal $\sum_i \partial/\partial u^i$ and $\sum_i u^i\partial/\partial u^i$ respectively, and the commutation relations of the vector fields $\mathcal{L}_n = -\sum_i (u^i)^n\,\partial/\partial u^i$ become especially clear.

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[3]In this case, the commutative algebras T_xM are semisimple for $x \in U$; the word semisimple arises for this reason, and not because of the semisimplicity of the Euler vector field.

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