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An introduction to the theory of local zeta functions, by Jun-ichi Igusa, American Mathematical Society, Providence, RI, and International Press, Cambridge, MA, 2000, xii + 232 pp., $45.00, ISBN 0-8218-2015-X

Jun-ichi Igusa's book *An introduction to the theory of local zeta functions* [13] is a substantial addition to the literature on zeta functions. Igusa is a prominent researcher on both the Archimedean and non-Archimedean sides of this theory; indeed, in the p-adic case these zeta functions and their generalizations are called "Igusa local zeta functions". These are complex valued functions and are quite different from other local zeta functions which have a longer history. In fact, the subject has developed into an independent theory only in the past twenty years or so, and some fascinating open questions remain. The book is distinguished not only by the depth and breadth of the material that it introduces but also by the care and thoroughness with which it does so.

The text is written keeping in mind students who have studied basic graduate-level algebra, real and complex analysis, and general topology. Its aim is to make the field of local zeta functions accessible to new researchers and to that end the book is self-contained. Archimedean and non-Archimedean (or p-adic) local fields are treated with equal weight and the philosophy is that all completions must be considered for the theory to be developed fully. The book combines new results of Igusa's with other results that were previously available only in original papers. In his discussion of the p-adic local zeta function, Igusa makes clear how examples led to conjectures and conjectures led to theorems. As an example of this process, the book ends by proving the theorems of J. Denef and D. Meuser [4], [5]. These proofs use Hironaka's resolution of singularities and the functional equation of the Weil zeta function over finite fields. (These are the only results used in the book without proof; instead they are explained through examples.)

This book covers everything needed to begin investigating local zeta functions. In fact, this reviewer has already used the book's examples of the embedded resolution for curves and Igusa's method of p-adic stationary phase to introduce students to these functions.

The local zeta functions dealt with in this book are defined using the following notation: let K be any completion of a number field and let $f(x)$ be a non-constant polynomial in n variables and coefficients in K with s a complex variable, $|\cdot|_K$ an absolute value on K, $\Phi(x)$ a Schwartz-Bruhat function or more intuitively a "good function" on K^n that forces the following integrals to converge, and dx a suitably normalized Haar measure on K^n. Then

$$Z_\Phi(s) = \int_{K^n} |f(x)|_K^s \Phi(x)\, dx$$

is called the *local zeta function*. The standard choices for $\Phi(x)$ are $\exp(-\pi {}^t x x)$ for $K = \mathbb{R}$, $\exp(-2\pi {}^t x \bar{x})$ for $K = \mathbb{C}$, and the characteristic function of O_K^n for K a p-adic field and O_K its ring of integers. When $\Phi(x)$ is a standard function, it is simpler to use the notation $Z(s) = Z_\Phi(s)$. If the real part of s is positive, then the

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local zeta function is absolutely convergent and defines a holomorphic function in the right half plane $\mathrm{Re}(s) > 0$.

In the Archimedean case, the distributions defined by these functions are called *complex powers*. At the 1954 Amsterdam International Congress, I. M. Gel'fand suggested the existence of a meromorphic continuation for complex powers. The general theory of local zeta functions really begins around 1970 when Bernstein and S. I. Gel'fand jointly [2] and Atiyah [1] independently proved the fundamental theorem of complex powers. This theorem states that $Z_\Phi(s)$ has a meromorphic continuation to the whole complex plane with poles in a finite number of arithmetic progressions of negative rational numbers. These proofs used Hironaka's resolution of singularities. Bernstein later proved the theorem again using his theory of Bernstein polynomials [3].

Around the same time, Igusa [8] also used Hironaka's resolution to prove the corresponding theorem in the non-Archimedean case, stating that not only does $Z_\Phi(s)$ have a meromorphic continuation to the whole complex plane but that if πO_K denotes the maximal ideal of O_K and q the number of elements in the finite residue field $O_K/\pi O_K$, then $Z_\Phi(s)$ is, in fact, a rational function of $t = q^{-s}$ and we can let $Z_\Phi(t) = Z_\Phi(s)$. For example, when $f(x, y) = y^2 - x^3$ [13, p. 171], then

$$Z(t) = (1 - q^{-1})(1 - q^{-2}t + q^{-2}t^2 - q^{-5}t^5)\Big/(1 - q^{-1}t)(1 - q^{-5}t^6).$$

In both the Archimedean and non-Archimedean cases, the main theorem holds more generally when $|\cdot|_K^s$ is replaced by any character of the multiplicative group of K.

A central problem has been to determine the poles of the local zeta function. The first proofs of the fundamental theorem above reveal that the candidate poles can be described by the numerical data, a pair of positive integers, associated with each of the finitely many exceptional divisors of the resolution of the hypersurface $f(x) = 0$.

In the Archimedean case, Bernstein's second proof of the meromorphic continuation used the existence of a polynomial $b_f(s)$ associated with $f(x)$ in such a way that its zeros directly determine the arithmetic progressions of poles of the zeta function [3]. Bernstein's theorem states that for K a field of characteristic 0 and $f(x)$ a non-zero polynomial in $K[x_1, \ldots, x_n]$, there exists a differential operator P in $K[s, x_1, \ldots, x_n, \partial/\partial x_1, \ldots, \partial/\partial x_n]$ and a monic polynomial $b(s)$ in $K[s]$ such that

$$P \cdot f(x)^{s+1} = b(s)f(x)^s$$

for s in $\mathbb{Z}$. For example, when $f(x, y) = y^2 - x^3$, $P = 1/27\, \partial^3/\partial x^3 + 1/6\, x\, \partial^3/\partial x \partial y^2 + 1/8\, y\, \partial^3/\partial y^3 + 3/8\, \partial^2/\partial y^2$, and $b(s) = (s + 1)(s + 5/6)(s + 7/6)$. In general, the $b(s)$ in Bernstein's theorem of smallest degree is unique and is called Bernstein's polynomial of $f(x)$, or simply a b-function. This b-function is denoted by $b_f(s)$. In the example above, $b(s)$ itself is known to be $b_f(s)$. Kashiwara has shown in general that the roots of $b_f(s)$ are negative rational numbers [14]. Bernstein's description of the poles of the Archimedean local zeta function gives a very complete explanation of them in terms of the zeros of $b_f(s)$. However, the Archimedean zeta function itself is not easy to compute. In fact, the meaning of "to compute" is not very clear. Therefore, the explicit form of $Z(s)$ given in the book for $f(x)$ a basic relative invariant of any regular prehomogeneous vector space over $\mathbb{C}$ in Sato's theory is remarkable [13, p. 91].

In contrast, the p-adic situation is tantalizing because more is known about the explicit form of the rational Igusa local zeta function and computing it clearly means finding its rational function. In fact, if the coefficients of $f(x)$ are in O_K, then

$$Z(t) = N(t) \Big/ \prod_i (1 - q^{-n_i} t^{N_i}),$$

in which n_i, N_i are the numerical data mentioned above and $N(t)$ is a polynomial in $\mathbb{Z}[q^{-1}, t]$ such that $N(1) = \prod(1 - q^{-n_i})$. The denominator above is a maximal denominator and in most cases many of the factors in the denominator will not appear in the final formula. This cancellation is expected in general but is well understood only when f is a polynomial in 2 variables [22],[19]. From explicit calculations of both $Z(s)$ and $b_f(s)$, the zeros of the b-function clearly seem to play a similar role in designating the poles of the zeta function. However, the direct connection in the Archimedean case (an integration by parts using the defining property of the b-function) does not work in the p-adic case. Loeser [18] has shown that the real poles of the Igusa local zeta function are roots of $b_f(s)$ when f is a polynomial in 2 variables.

There is no general formula for $b_f(s)$ or $Z(s)$. However, in the case of Sato's theory of irreducible, regular, prehomogeneous vector spaces (IRPVSs) where $f(x)$ is irreducible and a connected, irreducible matrix group G acts transitively on the complement of the hypersurface $f(x) = 0$, the $b_f(s)$ have been computed [21], [15]. More precisely, Sato's b-functions $b(s)$ have been computed and they are known to be equal to $b_f(s)$. In addition, the $Z(s)$ are also known for the invariant $f(x)$ in 24 of the 29 IRPVSs and many have very complicated numerators [13]. In fact, it is known that the real poles of $Z(s)$ are among the zeros of $b_f(s)$ in all 29 cases [16].

Based on the calculations in the IRPVS case along with other calculations, Igusa conjectured that in the p-adic case for a general polynomial $f(x)$, the real parts of the poles of $Z_\Phi(s)$ are roots of $b_f(s)$ and the order of each pole does not exceed the multiplicity of the root of $b_f(s)$. The set of real poles is conjectured to be contained in the set of roots – the sets are not always equal as can be seen from the example of $f(x, y) = y^2 - x^3$ above, where the root $s = -7/6$ is not a real pole of the zeta function. In his survey paper [12], Igusa proposes the problem of finding a formula for $b_f(s)$ and for $Z(s)$ in the IRPVS case (or even for a more general group invariant) in terms of the Dynkin diagram of G and the highest weight of the irreducibly represented group G. He also asks if $b_f(s)$ and $Z(s)$ can be described in terms of "invariants of the critical set of f" for a general $f(x)$. The numerator and denominator of the Igusa local zeta function reflect some deeper properties of the critical set of f. See [6], [12], and, of course, [13] for other open questions.

Not much is known about the numerators of $Z(s)$ and there are no general conjectures about them. However, there is a mystery about the numerators which surfaces in Igusa's book. The denominators are always simple products of the form explained above, but the numerators often contain complicated polynomials in $u = q^{-1}$ and $v = t$ and as polynomials in $\mathbb{Q}[u, v]$ they are absolutely irreducible. In particular, similar strange cubic polynomials with some interesting properties appear in the numerators of local zeta functions for polynomials which seem to be unrelated. The example below the next theorem gives a numerator involving the mysterious cubic. The zeros of these numerators in s will create some problems for the meromorphic continuation of the Euler product [17], [10].

The two important theorems which do contain information about the explicit form of $Z(s)$ were both proposed by Igusa as conjectures [9], [11]. He formulated them based on his explicit calculations. The proofs of these theorems are the finale of the book. The first is a theorem proved by Meuser [20] for polynomials that are non-degenerate with respect to their Newton polyhedra and then in general by Denef [4] and is stated as follows:

Theorem 1. *(Denef) If a homogeneous polynomial $f(x)$ with coefficients in O_K has good reduction modulo π, then*

$$\deg_t(Z(t)) = -\deg(f(x)).$$

For example, if $f(x)$ is the Freudenthal quartic, a homogeneous invariant of degree 4 of the 56 dimensional representation of a simple group of type E_7 and q is odd [13, pp. 178-183], then

$$Z(t) = (1-q^{-1})(1-q^{-14})C(t)/(1-q^{-1}t)(1-q^{-11}t^2)(1-q^{-19}t^2)(1-q^{-28}t^2)$$

where

$$C(t) = \{1+q^{-14} - q^{-11}(1+q^{-4}+q^{-8}-q^{-18})t + q^{-15}(1-q^{-10}-q^{-14}-q^{-18})t^2 + q^{-30}(1+q^{-14})t^3\}$$

and it is clear that the degree of the numerator of $Z(t)$ minus the degree of its denominator equals negative the degree of $f(x)$.

The second theorem is due to Denef and Meuser [5]:

Theorem 2. *(Denef-Meuser) If a homogeneous polynomial $f(x)$ with coefficients in O_K has a good reduction modulo π and there exists a rational function $Z(u,v)$ of two variables u, v such that if $Z_L(s)$ denotes the $Z(s)$ for the same $f(x)$ relative to a finite algebraic extension L of K, then*

$$Z_L(s) = Z(q_L^{-1}, q_L^{-s})$$

for all L, then the unique $Z(u,v)$ satisfies the functional equation

$$Z(u^{-1}, v^{-1}) = v^{\deg(f)} Z(u,v).$$

Actually, the Denef-Meuser theorem is more general; it is stated here as first conjectured by Igusa. The book proves the more general theorem. The simplicity of the functional equation is remarkable given complicated $Z(t)$ like the one for the Freudenthal quartic above, but the reader can easily verify it for that example.

This book has broad scope and meticulous self-sufficiency. To illustrate these qualities, I will give a rough outline of the material in each chapter.

The first chapter briefly reviews the background that the book assumes. The second section on Noetherian rings proves theorems on the minimal representation of an ideal and on local rings, including Nakayama's lemma. The third section proves Hilbert's theorems on polynomial ideals, including the basis theorem, the Nullstellensatz, and his theorems on characteristic functions.

Chapter 2 contains the proofs of the implicit function theorem for an arbitrary complete field (with additional features in the non-Archimedean case) and the Weierstrass preparation theorem. These theorems are then used to explain K-analytic manifolds and differential forms. In addition, theorems about the critical sets and critical values of polynomials are proved. The theorems in this chapter are essential in the desingularization process and for integration on manifolds, in general.

In Chapter 3, monoidal transformations and Hironaka's desingularization theorem are introduced. Hironaka's theorem is also explained for curves and two examples are explicitly worked out.

Bernstein's theory of D-modules, the Bernstein polynomial $b_f(s)$, and Bernstein's original proof of the main theorem in his theory are developed in Chapter 4.

Archimedean local zeta functions are handled in Chapter 5. First, the character groups of continuous homomorphisms $\mathrm{Hom}(K^\times, \mathbb{C}^\times)$ for $K = \mathbb{R}$ and $\mathbb{C}$ are determined. Next, the complete spaces of Schwartz functions and tempered distributions are established, and Gel'fand and Shilov's [7] method of analytic continuation is explained. Both Bernstein's proof of the meromorphic continuation of the local zeta function and the original proof using resolution are given. An application of the last theorem is the proof of the existence of an elementary solution of any partial differential equation with constant coefficients.

Chapter 6 treats Sato's theory of regular prehomogeneous vector spaces (RPVS) and the Sato b-function. It is shown that for $f(x)$ a basic relative invariant of the RPVS the Bernstein polynomial equals Sato's b-function and the zeros of Sato's b-function are rational. All properties of the Γ-function needed in the book are explained with proof so that the Archimedean local zeta functions can be worked out explicitly when $K = \mathbb{C}$ for any basic relative invariant and in a special case when $K = \mathbb{R}$.

Chapter 7 begins the non-Archimedean half of the book. Distributions in totally disconnected spaces are introduced as forming the dual space of the space of locally constant functions with compact support – the Schwartz functions in this setting. Next the case of a group acting on the totally disconnected space gives rise to a group action on these two spaces and eigendistributions are examined following Weil. The ground work for integration on p-adic manifolds and over fibers is laid and Serre's structure theorem for compact p-adic manifolds is proved.

Some of Tate's results on Haar integration for p-adic fields are explained before Chapter 8 introduces a slightly generalized local zeta function in the p-adic case. Igusa's influential proof of the rationality of the p-adic local zeta function is given. Weil's functions $F_\Phi(i)$ and $F_\Phi^*(i^*)$ are discussed along with their relationship to the zeta function. As a consequence, an asymptotic estimate of $F_\Phi^*(i^*)$ in terms of the poles of the zeta function is given. Finally, the theorems on eigendistributions are applied to give information on the poles of the generalized local zeta function for a group invariant f.

In Chapter 9, the classes of homogeneous polynomials for which the Igusa local zeta function will be computed in Chapter 10 are introduced. The treatment of quadratic forms is especially noteworthy. Witt's theorems on quadratic forms are given and the concept of a reduced quadratic form is introduced to allow a uniform treatment including the $\mathrm{char}(K) = 2$ case. Quadratic forms over finite fields are explained and the cardinality of the set $\{x \in \mathbb{F}_q^n \mid Q(x) = i\}$ is computed in all cases for a general reduced quadratic form Q. The cardinalities of classical groups over finite fields are worked out. Composition and Jordan Algebras are introduced. Their norm forms and the Freudenthal quartics are developed and, finally, an important identity first used by Gauss and its variants are proved.

Chapter 10 is crucial because it contains the evidence for many of the conjectures and theorems in the book. Numerous explicit examples of the calculation of the Igusa local zeta function are explained. The polynomials in Chapter 9 are all treated, and although most have been published in other papers, previous conditions on the characteristic of $\mathbb{F}_q$ have been removed in some cases. A p-adic stationary phase formula, a key lemma for working with group invariants, and a general integration formula for integrals of $\mathrm{SL}_n(O_K)$-invariant functions over $\mathrm{M}_{m,n}(O_K)$ are all presented.

Chapter 11 brings the theory together to prove Denef's theorem on the degree of the Igusa local zeta function and Denef and Meuser's theorem on its functional equation. Regular local rings, Hironaka's theorem in a more algebraic form, and Weil's zeta function over finite fields are all laid out here and then employed in the proofs.

To summarize, this book is written to draw researchers into the field of local zeta functions. It is self-contained and full of concrete examples. The writing is careful and the exposition is thorough and well organized. The book is not a textbook in the traditional sense. There are no exercises, for example. But it does have clear pedagogical goals. The reader is introduced to many intriguing problems and handed all the necessary tools from a wide variety of areas. These tools are then applied in the proofs of two important theorems. This work lays out Professor Igusa's personal view of the subject he has done so much to create and contains insights and reminiscences that could come only from him. It is the only introductory book on the subject of local zeta functions. In a succinct 232 pages, the theory and its unsolved problems become tantalizing lures.

References

[1] M. F. Atiyah, Resolution of singularities and division of distributions, Comm. Pure Appl. Math., 23 (1970), 145-150. MR **41:**815

[2] I. N. Bernstein and S. I. Gel'fand, Meromorphic property of the functions P^λ, Functional Anal. Appl., 3 (1969), 68-69. MR **40:**723

[3] I. N. Bernstein, The analytic continuation of generalized functions with respect to a parameter, Functional Anal. Appl., 6 (1972), 273-285. MR **47:**9269

[4] J. Denef, On the degree of Igusa's local zeta function, Amer. J. Math., 109 (1987), 991-1008. MR **89d:**11108

[5] J. Denef and D. Meuser, A functional equation of Igusa's local zeta functions, Amer. J. Math., 113 (1991), 1135-1152. MR **93e:**11145

[6] J. Denef, Report on Igusa's local zeta functions, Sém. Bourbaki 741 (1991), 1-25; Asterisque No. 201-203, 359-386. MR **93g:**11119

[7] I. M. Gel'fand and G. E. Shilov, Generalized functions. I, Academic Press (1964). MR **29:**3869

[8] J. Igusa, Complex powers and asymptotic expansions. I, J. Reine Angew. Math. 268/269 (1974), 110-130; II, ibid., 278/279 (1975), 307-321. MR **50:**254; MR **53:**8018

[9] J. Igusa, Some results on p-adic complex powers, Amer. J. Math., 106 (1984), 1013-1032. MR **86f:**11046

[10] J. Igusa, Zeta distributions associated with some invariants, Amer. J. Math., 109 (1987), 1-34. MR **89h:**11075

[11] J. Igusa, Universal p-adic zeta functions and their functional equations, Amer. J. Math., 111 (1989), 671-716. MR **91e:**11142

[12] J. Igusa, On local zeta functions, Amer. Math. Soc. Transl. (2) 172, (1996), 1-20. CMP 96:11

[13] J. Igusa, An Introduction to the Theory of Local Zeta Functions, AMS/IP Studies in Mathematics, 14, American Mathematical Society, Providence, RI; International Press, Cambridge, MA, 2000. CMP 2000:09

[14] M. Kashiwara, *B*-functions and holonomic systems (Rationality of *b*-functions), Invent. Math., 38 (1976), 33-53. MR **55:**3309

[15] T. Kimura, The *b*-functions and holonomy diagrams of irreducible regular prehomogeneous vector spaces, Nagoya Math. J., 85 (1982), 1-80. MR **84j:**32017

[16] T. Kimura, F. Sato, and X.-W. Zhu, On the poles of *p*-adic complex powers and *b*-functions of prehomogeneous vector spaces, Amer. J. Math., 112 (1990), 423-437. MR **91i:**11171

[17] N. Kurokawa, On the meromorphy of Euler Products (I), Proc. London Math. Soc. (3), 53 (1986), 1-47; (II), ibid., 209-236. MR **88a:**11084a; MR **88a:**11084b

[18] F. Loeser, Fonctions d'Igusa *p*-adiques et polynômes de Bernstein, Amer. J. Math., 110 (1988), 1-22. MR **89d:**11110

[19] D. Meuser, On the poles of a local zeta function for curves, Invent. Math., 73 (1983), 445-465. MR **85i:**14014

[20] D. Meuser, On the degree of a local zeta function, Comp. Math., 62 (1987), 17-29. MR **88i:**11093

[21] M. Sato and T. Kimura, A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya Math. J., 65 (1977), 1-155. MR **55:**3341

[22] L. Strauss, Poles of a two variable *p*-adic complex power, Trans. Amer. Math. Soc., 278 (1983), 481-493. MR **84k:**14019

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